A Theory of Capital Structure, Price Impact, and Long-Run Stock Returns under Heterogeneous Beliefs

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We study an environment with short-sale constraints and heterogeneous beliefs among outsiders and between insiders and outsiders. Firm insiders choose between equity, debt, and convertible debt to raise external financing. We analyze two settings: one in which heterogeneous beliefs is the only market imperfection and another in which there are significant security issue and financial distress costs. Our model generates a pecking order of external financing different from asymmetric information models, and new predictions for capital structure, sequential tranching of securities, the price impact of security issues, and long-run stock returns. We also provide a new rationale for convertible debt issuance. (*JEL* G32)

Introduction

Several authors have theoretically examined the stock price implications of heterogeneous beliefs and short-sale constraints on stock valuations. Miller (1977) argues that when investors have heterogeneous beliefs

about the future prospects of a firm, its stock price will reflect the valuation that optimists attach to it, because the pessimists will simply sit out of the market (if they are constrained from short selling). In another important paper, Morris (1996) shows that when divergence is greater in the valuations of the optimists and the pessimists, the current price of a stock in equilibrium is higher and hence the subsequent returns are lower. However, while the implications of heterogeneous beliefs among investors for capital markets have been examined at some length (see, e.g., Lintner 1969 for one of the earliest contributions), the corporate finance implications of these beliefs have not been adequately studied (with some notable exceptions that we will discuss later; see, e.g., Allen and Gale 1999).[1] The objective of this paper is to fill this gap by developing a theory of capital structure, the price impact (on equity) of security issuance, and the long-run stock returns following security issues in an environment of heterogeneous beliefs.

Several interesting questions arise in the above context. For example, does heterogeneity in beliefs between firm insiders and outsiders, and among outsiders, about the future prospects of a firm affect its security choice when raising external financing? Does a higher level of investor optimism result in its being more likely to issue equity over debt, or a combination of the two? Under which situations is it optimal to issue convertible debt? Can heterogeneity in beliefs explain the price impact of a firm's equity, debt, or convertible debt issue that traditional asymmetric information models cannot explain? Finally, how does heterogeneity in beliefs affect the long-run stock returns to issuers of equity, debt, and convertible debt? In particular, what explains the fact that, while the long-run stock returns of both equity and debt issuers have been empirically shown to be negative, the long-run stock returns of equity issuers are significantly more negative than those of debt issuers?

We answer these and other related questions in a heterogeneous beliefs framework.[2] The insiders of a firm, owning a certain fraction of equity in the firm, choose between equity, debt, or convertible debt to raise external financing to implement a positive net present value project. Market participants, each of whom have limited wealth, have heterogeneous beliefs about the long-run value of the firm. We can think of the average outsider belief as the level of "optimism" among outsiders, and the spread

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[1] We will discuss how our paper relates to the very small corporate finance literature making use of a heterogeneous beliefs assumption (e.g., Allen and Gale 1999) or an assumption of disagreement between firm insiders and outsiders (e.g., Dittmar and Thakor 2007) in Section 6.

[2] As in the existing literature on heterogeneous beliefs (see, e.g., Miller 1977; Morris 1996) we assume short-sale constraints throughout, so that the effects of differences in beliefs among investors are not arbitraged away. The above standard assumption is made only for analytical tractability: our results go through qualitatively unchanged as long as short selling is costly (see, e.g., Duffie, Gârleanu, and Pedersen 2002).

among outsider beliefs as the "dispersion" in their beliefs. The objective of firm insiders is to choose the security (or a combination of securities) to issue such that they maximize the long-run wealth of the firm's current shareholders, conditional on their own beliefs.

We first develop our theory of capital structure under heterogeneous beliefs by analyzing the firm's problem in our basic model where there are no market imperfections (i.e., no security issue costs or costs of financial distress) other than the above mentioned heterogeneity in beliefs. We first compare the case where the firm chooses between equity alone, debt alone, and convertible debt alone and characterize the optimal structure of these security issues. We show that, in the above setting, insiders of the firm will issue equity if and only if they expect the beliefs of the marginal outside investor to whom they will sell equity to be above their own beliefs about their firm's future prospects. This allows firm insiders to take advantage of outside investors' optimism and sell overvalued equity to them. On the other hand, if the marginal outside investor's belief is below insiders' own beliefs, they will choose to issue debt instead, taking advantage of the fact that the valuation of debt is relatively insensitive to outsider beliefs. We show that issuing convertible debt is never optimal in this setting, since it will be dominated by either equity alone (if the marginal outsider belief is above insider beliefs) or debt alone (if the marginal outsider belief is below insider beliefs).

We then analyze a firm's choice of issuing individual securities versus a combination of equity and debt to raise the required external financing. We show here that if the marginal outside investor (in the case of pure equity financing) is optimistic enough that his belief is above a certain threshold belief, the firm chooses to issue equity alone. If, however, the marginal investor's belief is below that threshold, the firm issues a combination of equity and debt, selling equity to the more optimistic outside investors and debt to the less optimistic ones. Further, the above implies that, the more optimistic or the more dispersed outsider beliefs are about the firm (or both), the more likely the firm is to issue equity alone rather than a combination of equity and debt. Finally, the greater the amount of external financing required by the firm, the lower the marginal investor's belief in the case of pure equity financing, and therefore, the more likely the firm is to use at least some debt to raise this financing. In our basic model, we also characterize the conditions under which a firm may undertake the sequential tranching of equity or debt issues: for example, rather than making a single equity issue, the firm makes two equity issues (at different valuations) within a short period of time.

Our full-fledged model incorporates a fixed cost of issuing each security (e.g., investment banking fees) and costs of financial distress into our basic model. In this full-fledged model, we first compare situations under which the firm chooses between issuing equity alone, debt alone,

and convertible debt alone. We show that, as in the basic model, issuing equity is optimal when the marginal outside investor's belief is above that of firm insiders. However, if the marginal investor's belief is below that of firm insiders, the firm issues either straight debt or convertible debt depending on the amount of external financing required. If this amount is small enough that, if the firm issues straight debt, there is no probability of default, then risk-free straight debt is the optimal choice of the firm. The intuition here is that, compared to equity or convertible debt, risk-free debt is not sensitive to outsider beliefs and does not suffer any undervaluation. If, however, the investment amount required is large enough that any straight debt issued incurs a positive probability of default, then the firm prefers to issue convertible debt rather than straight debt. This is because, while both risky straight debt and convertible debt will be undervalued to the same extent in this situation, issuing convertible debt with an appropriately chosen conversion ratio allows the firm to minimize expected costs of financial distress.

We then analyze a firm's choice of issuing individual securities versus a combination of equity and debt. We first show that, if the marginal outside investor is optimistic enough that his belief is significantly above firm insiders' beliefs, the firm will raise the required amount by issuing equity alone. If, however, the marginal outsider's belief is below firm insiders' beliefs, then the firm will find it optimal to issue a combination of equity and straight debt (risky or risk-free) if the issue costs involved are small. If the marginal investor's belief is above a certain threshold belief, the firm issues a combination of equity and risk-free debt; if the marginal investor's belief is below this threshold belief, the firm issues a combination of a smaller amount of equity and a large amount of (risky) debt. The threshold belief will depend on the firm's cost of financial distress. Finally, if the issue costs are large enough that issuing a combination of securities is significantly costly, the firm prefers to issue convertible debt instead of a combination of equity and straight debt. The advantage of selling convertible debt alone over selling a combination of equity and straight debt is that it reduces the firm's aggregate issue cost, but it has the disadvantage that the firm has to sell convertible debt at a uniform price to a single group of investors. Note that, in such a setting, issuing convertible debt alone will also dominate issuing straight debt alone, since it allows the firm to raise the same amount of external financing by offering a smaller face value than straight debt, thus reducing the firm's expected costs of financial distress as well.

Next, we study the price impact of equity, debt, and convertible debt issues, and study how the dispersion in investor beliefs affects the price impact of an equity issue. Note that, by price impact, we mean the abnormal return to the firm's equity from the price prevailing before the external security issue to the price prevailing after the issue date (*not* the

announcement date). Since the market is already aware that a security issue has been announced, one would expect the price impact to be zero in the absence of heterogeneity in investor beliefs.[3] We demonstrate that, in the presence of heterogeneous beliefs among outside investors, the price impact of an equity issue will be negative, while that of debt and convertible debt issues will be zero. The intuition for the fall in share price on the day of a new equity issue is that the marginal investor holding the firm's equity after the equity issue turns out to be less optimistic compared to the beliefs of the marginal investor holding the firm's equity prior to the equity issue, since, to sell additional equity to outsiders, the firm has to go down the belief ladder (i.e., it has to sell the new equity to outside investors who are less optimistic than those currently holding the firm's equity). Further, we show that the price impact of an equity issue will be more negative if the dispersion in outsider beliefs is greater. To the best of our knowledge, ours is the first model to generate predictions regarding the price impact of equity and debt issues.

Finally, we characterize the long-run stock returns of firms following equity, debt, and convertible debt issues. First, our analysis implies that the long-run stock returns after an equity issue will be negative. Second, it implies that the long-run stock returns after a (straight or convertible) debt issue will also be negative, but less negative on average than those following an equity issue. Finally, our analysis predicts that the long-run stock returns following an equity issue will be more negative if the dispersion in outsiders' beliefs is greater. The intuition behind the long-run negative stock returns following an equity issue is that, as additional information about the firm's operating performance becomes available to outside investors over time, the dispersion in outside investors' beliefs about the firm's prospects becomes smaller (as outside investors engage in Bayesian learning and update their heterogeneous priors based on this additional information, their beliefs become more homogeneous); further, the larger the initial dispersion, the larger the reduction in the dispersion in outsiders' beliefs with the arrival of new information. This reduction in dispersion means that the belief of the marginal investor holding the firm's equity will be lower after the arrival of new information compared to his belief at the time of the equity issue, thus leading to a reduction, on average, in the price of the firm's equity in the long run. Since the dispersion in outsider investors' beliefs when a firm (optimally) chooses to issue equity will be greater than in situations where it (optimally) chooses to issue straight debt or convertible debt (ceteris paribus),

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[3] In other words, asymmetric information models will not be able to generate a significant price impact for an equity issue, since there is no new information flow from firm insiders to outsiders on the day of an equity issue.

the long-run stock return following an equity issue will be more negative than that following a straight debt or a convertible debt issue.

It is worth noting that the above results on the relative magnitudes of the long-run stock returns following equity versus that following straight or convertible debt issues are unique to our model; they cannot be generated by asymmetric information models, for example. Thus, our model provides an explanation for the empirical regularity that the long-run stock returns following equity issues are more negative than those following debt issues for the first time in the literature.

The implications of our model have motivated a recent empirical study by Chemmanur, Michel, Nandy, and Yan (2011). They test some of the above implications of our model using measures of investor optimism developed by Baker and Wurgler (2006), and the two standard proxies for heterogeneity in investor beliefs used in the literature, namely, the dispersion in analyst forecasts and abnormal share turnover. Their findings are strongly consistent with the predictions of our model. First, as predicted by our model, they find that the probability of a firm issuing equity rather than debt is increasing in both the level of optimism of outside investors and the dispersion in outsider beliefs. Second, they find that, consistent with our model prediction, the price impact on a firm's equity is negative for an equity issue and zero for a debt issue (they find an average price impact of −2.8% around equity issues and zero percent around debt issues). These results are robust to controlling for the fact that the choice of security to issue (debt versus equity) is itself determined by the average level of outsider beliefs (optimism) and the dispersion in these beliefs. Third, they find that, while the long-run stock returns to both debt and equity issuers are negative, the stock returns to equity issuers are significantly more negative than those to debt issuers, again consistent with our model's predictions. Finally, they find that, the more optimistic outside investors are at the time of an equity issue and more dispersed their beliefs, the more negative the long-run (one and two year) stock returns are to the firm after equity issuance, which also supports our model's predictions.[4]

1. The Basic Model

There are three dates in the model: time 0, 1, and 2. At time 0, insiders of a firm own a fraction α of the firm's equity. The remaining $1 - \alpha$ is held

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[4] As Morris (1995) has argued in an important paper, differences in beliefs are quite consistent with rationality. Thus, in our setting, rational agents with heterogeneous priors "agree to disagree" about the future cash flows of the firm. In other words, our model develops a theory of security issuance and price impact in a setting of rational investors with heterogeneous beliefs and short-sale constraints. It is therefore able to generate many of the predictions claimed by the behavioral finance literature without resorting to the assumption of investors suffering from various behavioral biases.

by a group of outside shareholders. The total number of shares in the firm is normalized to one. At time 1, the firm needs to raise an amount of I from outside investors to fund the firm's project.[5] At time 2, the cash flows from the firm's project are realized and become common knowledge to all market participants, which can be either X^H or X^L, where $X^H > X^L > 0$.[6]

There is a continuum of investors in the market, with an aggregate wealth of $W > 0$. Each investor has the same amount of wealth. Market participants have heterogeneous beliefs about the future (time 2) cash flows of the firm. Firm insiders believe that with probability θ^f, the cash flow will be X^H, and with probability $1 - \theta^f$, the cash flow will be X^L. We assume that $\theta^f X^H + (1 - \theta^f) X^L > I$ so that firm insiders believe that the project has positive net present value. Potential (new) outside investors' beliefs about the value of the firm are uniformly distributed over the interval $[\theta^m - d, \theta^m + d]$.[7] We can think of θ^m as the "average" or "mean" belief of outsiders, and d as the dispersion in outsiders' beliefs (we will sometimes refer to θ^m as the level of "optimism" among potential outside investors). We use θ to index an agent whose belief is θ. Agent θ believes that with probability θ the firm's time 2 cash flow will be X^H, and with probability $(1 - \theta)$, the cash flow will be X^L.[8] Clearly, existing investors who already hold the firm's stock at time 0 will be the most optimistic outside investors, and their beliefs are greater than $(\theta^m + d)$. We assume that the existing outside shareholders holding the outstanding stock in the firm have already exhausted their wealth so that they cannot buy any additional securities newly issued by the firm at time 1.

The menu of securities available to the firm consists of common equity, straight debt, and convertible debt. In the basic model (Section 1), we assume that the firm does not incur any frictional cost of issuing securities (i.e., no issue or underwriting costs) or any deadweight cost of financial distress. Throughout the paper, we assume that all investors are subject to a short-sale constraint; that is, no short selling in the firm's security is

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[5] When outsiders' valuation of the new project is greater than that of firm insiders, it may be beneficial for the latter to sell equity that raises an amount larger than I to take advantage of the optimistic beliefs of outsiders with respect to the firm's new project. We assume here that the firm raises only the minimum amount required, I, to fund the firm's project due to considerations of corporate control or other reasons we do not model here. Modeling the optimal amount of external financing raised complicates our model considerably without changing the qualitative nature of our results.

[6] Note that the cash flows X^H and X^L are realized conditional on the project being financed and implemented.

[7] While we assume that outsiders' beliefs are uniformly distributed for analytical tractability, the qualitative nature of our results is unaffected by this assumption.

[8] Further, there are enough outsiders who believe that the project has positive net present value so that, for all securities among the menu of securities available to the firm, the marginal outside investor providing funding for implementing the project believes it to have net present value large enough that the firm insiders' participation constraint is satisfied (i.e., they are better off implementing the new project (than not implementing it) by selling that security to outsiders).

At time 0, insiders of a firm own a fraction α of the firm's equity. The remaining $1 - \alpha$ is held by a group of outside shareholders.

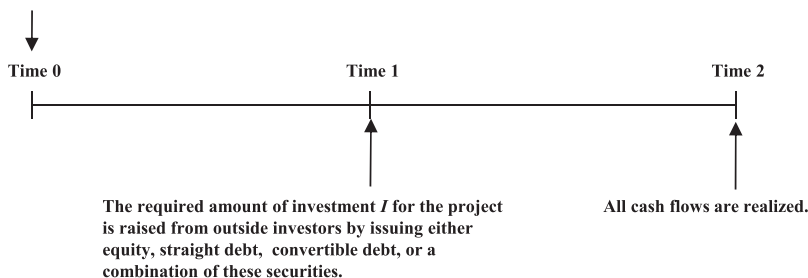**The total number of shares outstanding in the firm is normalized to one.**

Time 0   Time 1   Time 2

The required amount of investment I for the project is raised from outside investors by issuing either equity, straight debt, convertible debt, or a combination of these securities.   **All cash flows are realized.**

Figure 1
Sequence of events

allowed in the economy. We also assume that the amount of total wealth available to all investors is relatively large compared to the amount of money the firm wants to raise, so that $W > 2I$.[9] We assume that investors in the capital market suffer from a borrowing constraint, so that the amount W available to them for investment in the firm is inclusive of any amount that they are able to borrow.

The objective of firm insiders is to choose the optimal security to issue such that they maximize the expected time 2 payoff of current shareholders, based on firm insiders' belief, θ^f.[10] There is a risk free asset in the economy, the net return on which is normalized to zero. All agents are risk-neutral. Thus, firm insiders choose the optimal security, S, to maximize the following objective function

$$\max_{S} E_1[CF_2^{equity}|S, \theta^f], \tag{1}$$

where $E_1[CF_2^{equity}|S, \theta^f]$ is the time 1 expected value (according to firm insiders' belief) of the time 2 cash flows to the current equity holders of the firm, conditional on issuing security S, where S can be either equity, straight debt, or convertible debt. The sequence of events in the basic model is given in Figure 1.

1.1 The structures of individual security issues

In this subsection, we characterize and discuss the optimal structure of a security issue, assuming that the firm raises the required amount of

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[9] This is clearly an innocuous assumption, since, with very rare exceptions, the amount a firm wishes to raise in the capital market is small relative to the amount of capital available in the entire capital market.

[10] Since firm insiders hold a fraction α of the firm's shares, maximizing the value of current shareholders is equivalent to maximizing the value of shares held by firm insiders.

external financing by issuing equity alone (lemma 1), straight debt alone (lemma 2) or convertible debt alone (lemma 3). We first analyze the case in which the firm issues equity alone to outside investors, to raise the required amount of investment I at time 1.[11]

Lemma 1. (The structure of an equity issue)
When the issuing firm chooses to issue common stock alone to raise the amount of investment I, it has to issue a total of

$$E_1 = \frac{I}{\hat{\theta} X^H + (1 - \hat{\theta}) X^L - I} \qquad (2)$$

shares of new stock to outside investors at the price $PE_1^{Equity} = \hat{\theta} X^H + (1 - \hat{\theta}) X^L - I$, where the marginal investor in the firm's equity has the belief $\hat{\theta} = \theta^m + d\left(1 - \frac{2I}{W}\right)$ about the firm's cash flow at time 2. The equity price PE_1^{Equity} is decreasing in the amount of investment I.

Under heterogeneous beliefs and short-sale constraints, the firm will offer equity only to the most optimistic investors in the market. The (uniform) price at which the firm sells shares to outsiders depends on the belief of the marginal outside investor in the firm's equity, denoted by $\hat{\theta}$. This marginal investor is determined by starting with the most optimistic outside investor willing to invest in the firm (whose belief is given by $(\theta^m + d)$) and working down the ladder of outside investors' beliefs until the entire amount I required for investment in the firm is raised by selling equity. This means that the price of the firm's equity depends on two factors. The first factor is the average belief of investors in the market: the higher this average belief, the more optimistic the marginal investor's beliefs. The second factor that affects the price is the dispersion in outside investors' beliefs: holding the average belief constant, a higher dispersion in outside investors' beliefs means that the marginal investor's beliefs are more optimistic. Finally, when the amount of money the firm needs to raise from outsiders is higher, the firm needs to go lower down the belief ladder, and therefore the marginal investor who is holding the firms equity subsequent to the equity issue is less optimistic. Since the marginal investor is now less optimistic, the firms equity price is lower to reflect this, implying that a larger investment amount results in a lower equity issue price.

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[11] We assume that, in the case in which the firm raises its external financing through an equity issue, current shareholders do not participate in the issue, either as buyers or sellers. As discussed earlier, a wealth constraint will prevent current shareholders from buying any additional equity in the firm. We also assume that current shareholders are affiliated with firm insiders, and thus are prevented from selling into the equity issue (e.g., through lockup provisions).

We now assume that the firm issues straight debt alone to raise the required investment amount I. We normalize the face value of each unit of straight debt to one.

Lemma 2. (The structure of a straight debt issue)
When the issuing firm chooses to issue straight debt alone to raise the required amount of investment I:

1. If $I > X^L$, the firm issues risky straight debt. The price of each unit of debt is given by:

$$PD_1 = \frac{\hat{\theta}I}{I - (1 - \hat{\theta})X^L}. \tag{3}$$

The firm needs to issue a total of $F = \frac{I - (1 - \hat{\theta})X^L}{\hat{\theta}}$ units of straight debt to raise the required amount I, where the marginal investor in the firm's debt has the belief $\hat{\theta} = \theta^m + d\left(1 - \frac{2I}{W}\right)$ about the firm's cash flow at time 2.

2. If $I \leq X^L$, the firm issues risk-free straight debt. The price PD_1 of each unit of debt is one, and the firm needs to issue a total of $F = I$ units of straight debt to raise the required amount I.

When the firm issues straight debt alone to raise the required amount of new financing I, it raises these funds from the same group of investors as in the above case in which it issues equity alone. In other words, similar to an equity issue, the firm starts with the outside investor who is the most optimistic about the firm's future cash flows and works down the ladder of outsiders' beliefs until the entire amount I is raised by selling straight debt. Therefore, lemma 2 shows that the marginal investor in the firm's debt has the same belief $\hat{\theta} = \theta^m + d\left(1 - \frac{2I}{W}\right)$ as the marginal investor in its equity if the firm were to issue equity instead of debt alone (as in lemma 1).[12]

The price at which each unit of straight debt is sold by the firm, denoted by PD_1, is the price at which the marginal investor breaks even, given his belief $\hat{\theta}$. The firm issues F units of straight debt such that it is able to raise the entire investment amount I. One should note that in the case of risk-free debt, the security price is independent of the marginal investor's belief $\hat{\theta}$. However, in the case of risky debt, when the required amount of investment I is large, the debt price is also sensitive to the marginal outside investor's belief $\hat{\theta}$, though this sensitivity is much smaller than in the case of the price of equity.

We now analyze the case in which the firm issues convertible debt alone to raise the required amount of investment I. The terms of the convertible

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[12] One should note that, unlike an equity issue, the straight debt issue has no impact on the price of the firm's existing equity since the firm's marginal equity investor is the same as before the straight debt issue.

debt security are as follows: each unit has a face value of one and is sold at a price p at time 1; each unit of convertible debt can be converted into x shares of equity at time 2 if the investor chooses to exercise this option. We assume that there are restrictions on the conversion ratio x so that convertible debt will be a truly hybrid security between equity and straight debt (we specify these in lemma 3).

We normalize the number of shares of equity outstanding in the firm before it issues the convertible debt to one. To raise the amount I, the firm has to issue a total of I/p units of convertible debt. If investors decide to convert into equity at time 2, then the value of each unit of convertible debt from conversion is $\frac{x}{1+xI/p}V$, where V is the firm's market value at time 2, which is equal to either X^H or X^L. Investors will convert to common stock only if the payoff from conversion is greater than the face value of the convertible debt, 1, that is, if $\frac{x}{1+x\frac{I}{p}}V > 1$, or equivalently

$$\frac{V}{1 + x\frac{I}{p}} > \frac{1}{x}. \tag{4}$$

The quantity on the RHS of the inequality, $\frac{1}{x}$, is the conversion price of the convertible debt, whereas the LHS of the inequality corresponds to the firm value per share after the conversion. The following lemma characterizes the optimal conversion ratio x and the price p of the convertible debt, if the firm issues convertible debt alone to raise the required amount of investment financing I.

Lemma 3. (The structure of a convertible debt issue)
Let $x < \frac{\hat{\theta}X^H+(1-\hat{\theta})X^L}{X^L(\hat{\theta}X^H+(1-\hat{\theta})X^L-I)}$. Further, let $x > \frac{1}{X^H-I}$ if $I \leq X^L$, and $x > \frac{\hat{\theta}}{\hat{\theta}X^H+(1-\hat{\theta})X^L-I}$ otherwise.[13] If the firm decides to issue convertible debt alone to raise the required investment amount of I, then:

1. When outsiders are optimistic about the firm on average and their beliefs are more dispersed so that the marginal investor's belief $\hat{\theta}$ satisfies $\hat{\theta} = \theta^m + d\left(1 - \frac{2I}{W}\right) \geq \theta^f$, it is optimal for the firm to set the

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[13] These parametric restrictions ensure that the convertible debt is truly a hybrid of equity and straight debt. If the conversion ratio x is too high, new investors holding convertible debt will find it optimal to convert into equity at time 2 regardless of the value of the firm's cash flow. Thus, there will be practically no difference between convertible debt and equity. Similarly, if the conversion ratio x is too low, there will be practically no difference between convertible debt and straight debt. Thus, convertible debt will be a truly hybrid security between equity and straight debt, only if the conversion ratio x is between a lower bound and an upper bound. Existing shareholders can also impose an upper bound on the conversion ratio simply because of their concerns about maintaining control of the firm. Please see Appendix A for a numerical example on the optimal design of convertible debt in our setting.

conversion ratio at $x = \bar{x}$ given by (B28). In this case, the firm needs to issue

$$F = \frac{I}{\bar{p}} \tag{5}$$

units of convertible debt, where the convertible debt price $p = \bar{p}$ is given by (B30).

2. When outsiders are pessimistic about the firm on average and their beliefs are less dispersed so that the marginal investor's belief $\hat{\theta}$ satisfies $\hat{\theta} = \theta^m + d\left(1 - \frac{2I}{W}\right) < \theta^f$, it is optimal for the firm to set the conversion ratio at $x = \underline{x}$ given by (B24). In this case, the firm needs to issue

$$F = \frac{I}{\underline{p}} \tag{6}$$

units of convertible debt, where the convertible debt price $p = \underline{p}$ is given by (B27).

The marginal investor in the firm's convertible debt is determined by starting with the outside investor who is most optimistic about the firm's future cash flows and working down the ladder of outsider beliefs until the entire amount I required for investment in the firm is raised by selling convertible debt. Therefore, the belief of the marginal outside investor in the firm's convertible debt is identical to the belief $\hat{\theta}$ of the marginal investor in the above cases in which the firm issues equity or straight debt alone. Given the price p, the conversion ratio x, and the expected cash flows offered by each unit of the convertible debt, the marginal investor breaks even in return for his investment in the firm.

When outsiders are sufficiently more optimistic about the firm's future cash flows on average (i.e., the outsiders' average belief θ^m is higher) and their beliefs are more dispersed, the marginal outside investor with belief $\hat{\theta}$ also will be more optimistic about the firm's future cash flows than will firm insiders (i.e., $\hat{\theta} \geq \theta^f$). In this case, we show that it is optimal for firm insiders to set the conversion ratio x to the highest possible value $\bar{x}$ and thereby to maximize the equity component of the convertible debt. This makes sense since this equity component will be overvalued by the marginal outside investor relative to firm insiders' belief, and therefore, firm insiders will seek to benefit from capturing the outsiders' optimism on behalf of the existing shareholders by maximizing the equity component of convertible debt. The price of the

convertible debt in this case is given by Equation (5).[14] On the other hand, when outsiders are less optimistic about the firm's future cash flows on average, and their beliefs are less dispersed, the marginal outside investor also will be less optimistic about the firm's future cash flows than firm insiders. In this case, it is optimal for firm insiders to set the conversion ratio to the lowest possible value $\underline{x}$ to minimize the equity component of the convertible debt, since this component will now be undervalued relative to firm insiders' belief. The price of the convertible debt in this case is then given by Equation (6).[15]

1.2 The choice between equity, debt, and convertible debt alone or a combination of securities

We first assume that the firm has the choice of issuing either equity alone, debt alone, or convertible debt alone.[16] The following proposition characterizes the conditions under which the firm chooses to issue each security.

Proposition 1. (The choice between equity alone, straight debt alone, and convertible debt alone)
Let $\hat{\theta}X^H + (1 - \hat{\theta})X^L > I$ so that the firm's project has positive NPV based on the marginal outside investor's belief $\hat{\theta} = \theta^m + d\left(1 - \frac{2I}{W}\right)$. If the firm can issue only one type of security to raise the required amount of I for the project from outside investors, then:

1. The firm will choose to issue equity alone if outsiders are optimistic about the firm on average, and their beliefs are very dispersed so that the marginal outside investor is more optimistic than firm insiders, that is, if $\hat{\theta} > \theta^f$;

2. The firm will choose to issue straight debt alone if outsiders are pessimistic about the firm on average, and their beliefs are not so dispersed so that the marginal outside investor is less optimistic than firm insiders, that is, if $\hat{\theta} \le \theta^f$;

3. The firm will never choose to issue convertible debt since convertible debt will be dominated by either equity alone or straight debt alone, depending on outsiders' beliefs.

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[14] However, we will later show in proposition 1 that if the firm is unconstrained with regard to its choice of security, so that it can choose among equity, straight debt, and convertible debt, it will always choose to issue equity under this scenario rather than to issue convertible debt, since equity will be even more overvalued than convertible debt in this situation.

[15] One should again note that, unlike an equity issue, the convertible debt issue has no impact on the price of the firm's existing equity since the firm's marginal equity investor remains the same before and after the convertible debt issue.

[16] Throughout the paper, we assume that, if the firm issues convertible debt, it is optimally designed from the firm's point of view, and that its design satisfies the parametric restrictions specified in lemma 3.

As discussed earlier, in each case, we showed that the marginal outside investor has the same belief $\hat{\theta}$ about the firm's future cash flow at time 2; that is, $\hat{\theta} = \theta^m + d\left(1 - \frac{2I}{W}\right)$, regardless of the particular security the firm chooses to issue at time 1. However, since each security has its own unique payoff structure depending on the state of the world at time 2, the expected payoffs of insiders and existing shareholders will be different across all three different securities.

In the case in which outside investors are more optimistic about the firm's future cash flows on average, that is, the average outsider belief θ^m is relatively high, and their beliefs are more dispersed, the belief of the most optimistic new investor in the firm's security (given by $(\theta^m + d)$) is likely to be significantly higher than that of firm insiders, that is, θ^f. Then, starting with this most optimistic investor willing to invest in the firm and working down the ladder of outsider beliefs, the belief of the marginal outside investor, $\hat{\theta}$, also should be more likely to be above that of firm insiders. In this situation, all these securities (equity, straight debt, or convertible debt) will be overvalued relative to firm insiders' belief. However, since equity is the most sensitive security to outsider beliefs, it also will be the most overvalued security based on insiders' beliefs if the marginal outside investor is more optimistic than firm insiders.[17] Therefore, in this scenario, we show that the firm chooses to issue equity alone instead of the other two securities to best capture outside investors' optimism.

On the other hand, when outside investors are more pessimistic about the firm's future cash flows, on average, and their beliefs are less dispersed, the belief of the most optimistic outside investor will not be as optimistic as in the scenario discussed in the previous paragraph. In this case, if the marginal investor's belief, $\hat{\theta}$, is below that of firm insiders, and the firm chooses to sell equity, its equity will be substantially undervalued relative to the insiders' belief. Therefore, the firm will choose to issue straight debt since this security is less sensitive to outsider beliefs than either equity or convertible debt, and therefore is the least undervalued.

The above proposition shows that, in the absence of issue costs and costs of financial distress, issuing convertible debt is never optimal for the firm in either of the above two scenarios. When the marginal outside investor is more optimistic than firm insiders, that is, $\hat{\theta} \geq \theta^f$, the equity component of convertible debt will be overvalued. However, in this case, firm insiders would be even better off by issuing common equity instead

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[17] Note that if we rank each security based on its value sensitivity to outsiders' beliefs about the firm's future cash flows, equity is the most sensitive security, since its payoffs are perfectly positively correlated with the state of the world. Straight debt is the least sensitive security to investor beliefs, since it promises the repayment of a fixed face value F unless the firm defaults in the future. Convertible debt, which is a hybrid of straight debt and equity, ranks in between the two with respect to its price sensitivity to outsider beliefs.

of issuing convertible debt with an overvalued equity component, and insiders can capture outside investors' optimism better by issuing equity rather than convertible debt.

On the other hand, when the marginal outside investor is more pessimistic than firm insiders, that is, $\hat{\theta} < \theta^f$, the equity component of convertible debt will be undervalued. In this case, while firm insiders are better off issuing convertible debt rather than equity (since the undervaluation of equity is more severe than that of convertible debt), they are even better off by issuing straight debt rather than convertible debt. Since straight debt always promises the repayment of a fixed face value no matter how good the state of the world, its undervaluation based on insiders' belief will be less severe than that of convertible debt.

We now consider the possibility that the firm can issue a combination of debt and equity to raise the necessary financing for its project.

Proposition 2. (The choice between equity alone, straight debt alone, convertible debt alone, and a combination of straight debt and equity)
Let $\theta^f < \theta^m + d$.

1. The firm will choose to issue equity alone if outsiders are very optimistic about the firm on average, and their beliefs are very dispersed so that the marginal outside investor's belief $\hat{\theta}$ is above the upper threshold belief θ_1, that is, $\hat{\theta} \geq \theta_1$.

2. The firm will choose to issue a combination of risk-free straight debt and equity if outsiders are moderately optimistic about the firm on average, and their beliefs are moderately dispersed so that the marginal outside investor's belief $\hat{\theta}$ is between the lower threshold belief θ_2 and the upper threshold belief θ_1, that is, $\theta_2 \leq \hat{\theta} < \theta_1$.

3. The firm will choose to issue a combination of risky straight debt and equity if outsiders are pessimistic about the firm on average, and their beliefs are not very dispersed so that the marginal outside investor's belief $\hat{\theta}$ is below the lower threshold belief θ_2, that is, $\hat{\theta} < \theta_2$.

4. It is never optimal for the firm to issue straight debt alone.

5. The firm will never issue convertible debt since it is always dominated by a combination of straight debt and equity.

When the average outside investor is very optimistic about the firm's future cash flows and outsiders' beliefs are very dispersed, the marginal outside investor will be willing to pay a relatively high price for the firm's equity with respect to the insiders' beliefs. In this case, the above proposition shows that it is optimal for the firm to issue equity alone to capture the high degree of optimism of the marginal outside investor. Issuing equity alone in this case also dominates issuing a combination of debt

and equity because of the following trade-off the firm faces when issuing a combination of debt and equity. While raising part of the total funding I through debt issuance will increase the equity price (since less money is raised through equity issuance), the debt price will not be as sensitive to the optimism in outsiders' beliefs as the equity price. When the marginal outside investor has a very optimistic view of the firm even in the case in which the entire amount of funding is raised by issuing equity, issuing equity alone better captures the optimism of outside investors than issuing a combination of equity and debt. Thus, firm insiders will choose to maximize the overvaluation benefit they capture due to the large difference in equity valuation between insiders and the marginal outside investor.

When the average outside investor is not so optimistic about the firm's future, and outsiders' beliefs are not so dispersed, issuing equity alone to raise the entire funding will hurt the firm's existing shareholders (and insiders), if the marginal outside investor has a lower valuation of the firm than do the insiders. Similarly, if the marginal outside investor's valuation of the firm is only slightly higher than the insiders' valuation of the firm (assuming that the firm issues equity alone), the firm actually can be better off by raising part of the total funding I through debt and thereby can increase the equity price paid by the marginal equity investor. In such cases, the above proposition shows that it is optimal for the firm to issue a combination of debt and equity to raise the required funding I for the firm's project. Starting with the most optimistic outside investor with belief $(\theta^m + d)$ and going down the ladder of outsider beliefs, the firm can raise some money $(I - I_D)$ by issuing equity to the most optimistic investors and the rest (I_D) by issuing debt to the less optimistic investors until the entire amount of I is raised. In this way, as long as the most optimistic outside investor is more optimistic than the firm insiders, that is, $\theta^f < (\theta^m + d)$, the firm can still capture and benefit from the optimism of the most optimistic outsiders by issuing some equity. On the other hand, by issuing some debt simultaneously, the firm will not be hurt by the views of the less optimistic and downright pessimistic outside investors.

The above proposition shows that, to raise a given level of required investment funding I, the firm prefers to issue equity alone if the marginal outside investor's belief $\hat{\theta}$ exceeds the threshold value of θ_1, which may be above the insiders' belief θ^f. This condition will be satisfied when the average outside investor is very optimistic about the firm's future cash flows (θ^m is high relative to θ^f) and outsiders' beliefs are very dispersed (the dispersion in outsiders' belief d is large). As the average optimism of outsiders θ^m and/or the dispersion in their beliefs d decrease, the marginal outside investor becomes less optimistic. Hence, the cost of issuing undervalued equity increases, and the firm chooses to issue some amount of

debt (I_D) in combination with selling equity to reduce this undervaluation cost. As long as the marginal outsider investor is moderately optimistic (i.e., $\theta_2 \leq \hat{\theta} < \theta_1$), the size of the debt issue will be small, and the firm will choose to issue a combination of risk-free debt ($I_D \leq X_L$) and equity. However, if the marginal outsider is sufficiently pessimistic (i.e., $\hat{\theta} < \theta_2$), the firm will increase the size of its debt issue, and choose to issue a combination of risky debt ($I_D > X_L$) and equity to strike the optimal balance between the firm's objective to reduce the cost of issuing undervalued equity to pessimistic outsiders by selling them some debt and its objective of capturing the optimism of the most optimistic outsiders by selling them some equity.

If it is feasible for the firm to issue a combination of straight debt and equity, issuing straight debt alone is never optimal since this fails to capture the optimism of those investors with very optimistic beliefs about the firm. If there exist some very optimistic outside investors who value the firm higher than the insiders, the firm can benefit from the optimism of these outsiders by issuing some equity to them. Thus, even if the average outside investor is not so optimistic about the firm's future prospects, issuing a combination of equity and straight debt dominates issuing straight debt alone as long as there exists some heterogeneity in outsiders' beliefs and the most optimistic outside investor is more optimistic than the firm insiders.[18]

When the firm issues a combination of straight debt and equity, it can sell equity to the most optimistic outside investors at a relatively high price and sell straight debt to the less optimistic outsiders. In contrast, when the firm issues convertible debt, the equity component and the debt component of the convertible security have to be sold to the same group of investors at a uniform price.[19] Thus, when the firm issues convertible debt, it is unable to achieve the optimal price differentiation between its debt and equity components. Therefore, in the absence of issuance costs, convertible debt is always dominated by a combination of straight debt and equity.

1.3 The sequential tranching of securities

In this subsection, we allow for the firm to tranche one of the two securities it issues: tranching of equity or tranching of straight debt

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[18] However, this particular result is true only under the assumption that there are no issue costs. When issue costs are significant (as we assume in later sections), we will show that it can be optimal for the firm to issue debt alone as well as equity alone under certain conditions.

[19] The marginal outside investor who is pricing the equity component of convertible debt is the same marginal investor who is pricing the debt component of it, so that both components are priced by the marginal investor with belief $\hat{\theta}$. However, if the firm instead issues a combination of straight debt and equity, the marginal equity investor with belief $\hat{\theta}_E = \hat{\theta} + \frac{2dI_D}{W}$ is willing to pay a higher price than the marginal convertible debt investor with belief $\hat{\theta}$.

(but not the tranching of both securities simultaneously, for analytical tractability). We assume that in the event of the tranching of a security, only two tranches will be issued. By "tranching" we mean that the firm makes sequential security offerings, where the firm does not raise its full financing requirement (to be raised from issuing that particular security) in a single security issue, but may split the security issue into two sequential security offerings. Further, we allow the firm to issue a combination of equity and debt (if that is optimal), where one of these security issues is split into two sequential tranches, while the other is "untranched."[20]

We first analyze the case in which the firm may issue a combination of two tranches of equity and untranched debt (i.e., only tranching of equity is allowed). If we let I_D denote the total amount of untranched debt raised to finance the project, then the amount of total equity raised in two tranches is equal to $I_E = I - I_D$. The advantage of issuing two equity tranches is that, when selling its equity, it helps the issuing firm further price-differentiate across outside investors with heterogeneous beliefs. We denote the amount of the first equity tranche that is sold to the most optimistic group of outside investors as I_1. The belief of the marginal investor in this equity tranche offering is then given by:

$$\hat{\theta}_1 = \theta^m + d\left(1 - \frac{2I_1}{W}\right). \tag{7}$$

The amount of the second equity tranche sold to the less optimistic group of outsiders is $I_2 = I_E - I_1$. Going down the ladder of outsider beliefs, the belief of the marginal investor in this equity tranche offering is then given by:

$$\hat{\theta}_E = \theta^m + d\left(1 - \frac{2I_E}{W}\right). \tag{8}$$

For any given amount of total equity offered, I_E, the firm determines the optimal breakpoint of equity tranching by choosing that level of I_1 that minimizes the share dilution of the firm's existing shareholders. In other words, the firm insiders' objective is to maximize the fraction of equity held by the firm's existing shareholders after the two new equity issues. After having determined the optimal I_1 conditional on I_E, the firm then determines the optimal amount of total equity issued I_E. If $I_D \leq X^L$,

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[20] For analytical simplicity, we exogenously assume that the firm can issue only two sequential equity or debt tranches. Since, in this basic model, we assume that there are no issue costs associated with each security issue, the firm may benefit from issuing more than two tranches. However, in practice, the optimal number of tranches may be determined by a trade-off between aggregate issue costs and the benefit from tranching.

the debt issued by the firm is risk-free. In this case, the firm insiders solve the following problem:

$$\max_{I_E, I_1} \left(1 - \frac{I_1}{X_{\hat{\theta}_1} - (I - I_E)} - \frac{I_E - I_1}{X_{\hat{\theta}_E} - (I - I_E)} \right) (\theta^f X^H + (1 - \theta^f) X^L - (I - I_E)),$$

(9)

subject to the constraint $I - X^L \leq I_E \leq I$, and where $X_{\hat{\theta}_1} = \hat{\theta}_1 X^H + (1 - \hat{\theta}_1) X^L$ and $X_{\hat{\theta}_E} = \hat{\theta}_E X^H + (1 - \hat{\theta}_E) X^L$. If $I_D > X^L$, then the debt issued by the firm is risky, and the firm insiders solve the following problem:

$$\max_{I_E, I_1} \left(1 - \frac{I_1}{\hat{\theta}_1 (X^H - F)} - \frac{I_E - I_1}{\hat{\theta}_E (X^H - F)} \right) \theta^f (X^H - F),$$

(10)

subject to the constraint $0 < I_E \leq I - X^L$, and where the face value of the debt is $F = \frac{I - I_E - (1 - \hat{\theta}) X^L}{\hat{\theta}}$.

Proposition 3. (Sequential tranching of equity)
When tranching of equity is allowed, the firm always prefers issuing a package of two tranches of equity and untranched debt compared to issuing a package of untranched equity and untranched debt for any given amount of debt issued, I_D. Further,

1. The firm will issue two tranches of equity alone to finance the project, that is, $I_E^* = I$, if the marginal investor is very optimistic so that $\hat{\theta} \geq \theta_1$. In this case, the optimal breakpoint of equity tranching I_1^* is given by (B61).
2. If the marginal investor with belief $\hat{\theta}$ is moderately optimistic so that the firm finds it optimal to issue a combination of two tranches of equity and risk-free debt to finance the project, that is, $I - X^L \leq I_E^* < I$, the optimal breakpoint of equity tranching I_1^* is given by (B60). The optimal amount of total equity issued I_E^* is implicitly defined by (B63).
3. The firm will choose to issue a combination of two tranches of equity and risky debt, that is, $0 < I_E^* < I - X^L$, if the marginal outside investor's belief is below a lower threshold belief θ_e^t given by (B72), that is, if $\hat{\theta} < \theta_e^t$. In this case, the optimal breakpoint of equity tranching I_1^* is given by (B67). The optimal amount of total equity issued I_E^* is given by (B70).

This proposition shows that, for a given amount of untranched debt issued I_D, the firm benefits from its ability to further price-differentiate across outside investors by issuing two sequential tranches of equity. If the firm sells all its equity issue in a single tranche, the equity price will be

determined by the marginal equity investor with belief $\hat{\theta}_E$ only. In the case of tranched equity, however, the firm is able to offer its first equity trance (worth I_1) to a more optimistic group of outside investors with marginal belief $\hat{\theta}_1$. Since $\hat{\theta}_1 > \hat{\theta}_E$, this means that the first equity tranche is offered at a strictly higher price than the second equity tranche, which is offered to a less optimistic group of outside investors with marginal belief $\hat{\theta}_E$.[21] When the firm determines its optimal breakpoint of equity tranching, I_1^*, its trade-off is between the size of the first equity tranche and its price. If I_1 is too small, the price of the first equity tranche will be much higher than the price of the second equity tranche, but only a small number of shares will be offered at this high price. On the other hand, if I_1 is too large, the price of the first tranche will be only slightly higher than the price of the second tranche. The optimal breakpoint I_1^* maximizes the fraction of equity held by the firm's existing shareholders after the issuance of new securities.

Next, we analyze the case in which the firm may issue a combination of untranched equity and two tranches of debt (i.e., only tranching of debt is allowed). If we let I_D denote the total amount of debt raised to finance the project, then the amount of equity raised is equal to $I_E = I - I_D$. The belief of the marginal investor in the firm's equity is then given by:

$$\hat{\theta}_E = \theta^m + d\left(1 - \frac{2I_E}{W}\right). \tag{11}$$

If $I_D \leq X^L$, the firm issues only one tranche of risk-free debt. Thus, the firm can issue two tranches of debt if and only if $I_D > X^L$, which is equivalent to $0 \leq I_E < I - X^L$. In this case, we denote the amount of financing raised by the riskier debt tranche as I_1. The belief of the marginal investor in this debt tranche is then given by:

$$\hat{\theta}_1 = \theta^m + d\left(1 - \frac{2(I_E + I_1)}{W}\right). \tag{12}$$

Finally, the amount of financing raised by the safer debt tranche is $I_2 = (I_D - I_1)$, and the marginal investor in this debt tranche is equal to $\hat{\theta} = \theta^m + d\left(1 - \frac{2I}{W}\right)$.

For any given amount of total debt I_D, the firm determines the optimal breakpoint of debt tranching by choosing that level of I_1 that minimizes the total face value promised by the two debt tranches. If $(I_D - I_1) \leq X^L$, then the safer debt tranche will be completely risk-free, and the face

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[21] For either equity or debt issues, we rule out any "strategic waiting" on the part of the optimistic investors, in the hope that the firm may issue a lower priced tranche of that security. In practice, this seems to be an innocuous assumption, since it is reasonable to expect investors to have some uncertainty about future security issues, so that they will invest in a security if its offer price is weakly lower than their valuation of this security (based on their own beliefs about the firm's future cash flows).

values F_1 and F_2 of the debt tranches will be determined by the following two equations, respectively:

$$I_1 = \hat{\theta}_1 F_1 + (1 - \hat{\theta}_1)(X^L - (I_D - I_1)), \qquad F_2 = I_D - I_1. \qquad (13)$$

In this case, it is straightforward to show that it is optimal to set $I_1 = I_D - X^L$ so that $F_2 = X^L$ and $F_1 = \frac{I_D - X^L}{\hat{\theta}_1}$. On the other hand, if $(I_D - I_1) > X^L$, then the safer debt tranche also will be risky, and the face values F_1 and F_2 of the debt tranches will be determined by the following two equations, respectively,

$$I_1 = \hat{\theta}_1 F_1, \quad I_D - I_1 = \hat{\theta} F_2 + (1 - \hat{\theta}) X^L. \qquad (14)$$

The equity value of the firm will be $\hat{\theta}_E(X^H - F)$. Thus, in case the firm issues a combination of untranched equity and two tranches of debt, the firm insiders solve the following problem:

$$\max_{I_E, I_1} \left(1 - \frac{I_E}{\hat{\theta}_E(X^H - F)}\right) \theta^f (X^H - F), \qquad (15)$$

subject to the constraints (a) $0 \le I_E < I - X^L$, (b) $0 \le I_1 \le I - I_E$, and (c) $F = F_1 + F_2$.

Proposition 4. (Sequential tranching of debt)
Let $\theta^f < \theta^m + d$. When tranching of debt is allowed, the firm's optimal policy of security issuance is as follows:

1. The firm will choose to issue equity alone if the marginal outside investor's belief $\hat{\theta}$ is above the upper threshold belief θ_1^t, that is, $\hat{\theta} \ge \theta_1^t$.
2. The firm will choose to issue a combination of risk-free straight debt and untranched equity if the marginal outside investor's belief $\hat{\theta}$ is between the threshold belief θ_2^t and the upper threshold belief θ_1^t, that is, $\theta_2^t \le \hat{\theta} < \theta_1^t$.
3. The firm will choose to issue a combination of one tranche of risk-free debt, one tranche of risky debt, and untranched equity if the marginal outside investor's belief $\hat{\theta}$ is between the lower threshold belief θ_3^t and the threshold belief θ_2^t, that is, $\theta_3^t \le \hat{\theta} < \theta_2^t$.
4. The firm will choose to issue a combination of two tranches of risky debt and untranched equity if the marginal outside investor's belief is below the lower threshold belief θ_3^t, that is, $\hat{\theta} < \theta_3^t$.

This proposition shows that if the marginal investor with belief $\hat{\theta}$ is sufficiently optimistic, that is, if $\hat{\theta} \ge \theta_2^t$, the firm still prefers to issue equity alone (if $\hat{\theta} \ge \theta_1^t$) or a combination of equity and risk-free debt (if $\theta_2^t \le \hat{\theta} < \theta_1^t$). However, if outsiders are pessimistic on average and

the dispersion in their beliefs is low so that the marginal investor's belief $\hat{\theta}$ is below the threshold belief θ_2^t, the firm finds it optimal to issue a combination of untranched equity and two tranches of debt, where at least one of the debt tranches is risky. In this case, when issuing debt, the firm benefits from its ability to further price-differentiate across outside investors by issuing two tranches of debt. The riskier debt tranche worth I_1 is offered to a group of more optimistic outside investors with marginal belief $\hat{\theta}_1$, whereas the safer debt tranche, which is worth $I_2 = I_D - I_1$, and promises to pay X^L in the low cash flow state, is targeted to the most pessimistic group of outside investors. If $\theta_3^t \leq \hat{\theta} < \theta_2^t$, the safer debt tranche is completely risk-free. However, if the marginal investor with belief $\hat{\theta}$ is very pessimistic so that $\hat{\theta} < \theta_3^t$, the total amount of debt to be issued, I_D, will be significantly large, and even the safer debt tranche will be risky.

2. The Full-Fledged Model with Costs of Financial Distress and Issue Costs: The Choice between Equity, Straight Debt, and Convertible Debt

In this section, we introduce two costs into our model: issue costs and costs of financial distress. We denote the cost of issuing a security (e.g., underwriting fees) by C^I, and the firm's cost in the event of financial distress by C^B.[22,23] We analyze how these costs interact with heterogeneous investor beliefs in determining the firm's optimal choice of external financing among three different securities: equity, straight debt, and convertible debt.

2.1 The choice between equity alone, straight debt alone, convertible debt alone, and a combination of straight debt and equity

First, we analyze the case of prohibitively expensive issue costs in which the firm can issue only one type of security to finance its investment. We assume that, if the face value F of a debt security (straight or convertible) is strictly greater than the firm's cash flow X^L in the bad state, the firm faces a financial distress cost $C^B > 0$ when the low cash flow X^L is realized, since the firm has to default on a fraction of its promised payment F

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[22] Clearly, there are significant issue costs in practice. In the case of IPOs, the underwriting spread alone is 7% of the amount raised, with total issue costs between 10%–20% depending on the size of the issue. While underwriting spreads are generally lower for seasoned equity issues, they are nevertheless a significant percentage of the amount raised. For simplicity, we assume here that issue costs are constant (independent of the amount raised), and are the same for across debt and equity issues. Modeling issue costs more realistically (e.g., as the sum of a fixed component and a variable component that depends on the amount raised), and allowing the issue costs for equity issues to be greater than those for debt issues, will not change our results qualitatively (while adding complexity to our analysis).

[23] Such financial distress costs are either direct costs of bankruptcy or indirect costs of financial distress arising from distortions in managerial incentives (resulting in risk-shifting or underinvestment) as the firm approaches a high probability of default. Our analysis allows for the fact that such distress costs can vary significantly across industries, from very small in some industries, to moderate or large in others.

in this scenario. Otherwise, if $F \leq X^L$, the firm's cost of financial distress is 0. The following proposition characterizes the conditions under which the firm issues equity alone, debt alone, or convertible debt alone.

Proposition 5. (The choice between equity alone, straight debt alone, and convertible debt alone)

1. If outside investors are optimistic about the firm's future cash flows on average, and their beliefs are widely dispersed so that the marginal outside investor is more optimistic than firm insiders, that is, if $\hat{\theta} \geq \theta^f$, it is optimal for the firm to issue equity alone.
2. If outside investors are pessimistic about the firm's future cash flows on average, and their beliefs are not so dispersed so that the marginal outside investor is less optimistic than firm insiders, that is, if $\hat{\theta} < \theta^f$, the firm's optimal security choice is as follows:
   a) If the required investment amount I is small so that $I \leq X^L$, it is optimal for the firm to issue risk-free straight debt.
   b) If the required investment amount I is large so that $I > X^L$, it is optimal for the firm to issue convertible debt with total face value $F = X^L$.

The intuition behind the above proposition is as follows. When the average outside investor is much more optimistic about the firm's future cash flows than are firm insiders, and outsiders' beliefs are very dispersed, the marginal outside investor will be more optimistic than are firm insiders. In this situation, it is optimal for the firm to issue equity alone. In this case, equity dominates both straight debt and convertible debt from the point of view of firm insiders since it best allows the firm to take advantage of the optimism among outsiders and thus to sell a security that is most overvalued relative to firm insiders' valuation conditional on their own belief. Furthermore, issuing equity alone allows the firm to avoid costs of financial distress.

In contrast, when the average outside investor is not so optimistic or downright pessimistic about the firm's future prospects and the dispersion in outsider beliefs is low (so that the marginal outside investor is less optimistic than firm insiders), equity is no longer the optimal security to issue. This is because, in this case, equity (or any other security with an equity component) will be undervalued relative to the belief of firm insiders. The firm will then issue either straight debt or convertible debt depending on the size of the required investment I.

In the presence of costs of financial distress, the choice between straight debt and convertible debt depends on the following trade-off. On the one hand, convertible debt has an embedded equity component, which will be undervalued relative to firm insiders' beliefs in this situation

(unlike straight debt, whose valuation is less insensitive to outsider beliefs): we call this the "undervaluation effect." On the other hand, the option to convert to equity embedded in convertible debt is also valuable, since it reduces the face value of the debt to be offered to outsiders in return for a given amount of financing, thereby reducing the probability of the firm going into financial distress (and consequently the expected financial distress cost incurred by the firm): we call this the "embedded option effect." In particular, the firm can set the face value of convertible debt such that it avoids bankruptcy with probability 1.

If the size of the investment is small, that is, $I \leq X^L$, so that the firm can raise this amount by issuing risk-free straight debt, it will incur no costs of financial distress. If the firm issues risk-free convertible debt alone instead, it will be undervalued with respect to risk-free straight debt, whose valuation is not sensitive to outsider beliefs. Therefore, in this case, the undervaluation effect will dominate the embedded option effect, and the firm will prefer to issue risk-free straight debt rather than convertible debt.

If the size of the required investment is large, that is, $I > X^L$, the firm cannot issue risk-free straight debt alone to raise this entire amount I. In this case, the embedded option effect of the convertible debt will favor issuing convertible debt to issuing risky straight debt. As we showed in lemma 2, risky straight debt is sensitive to the marginal investor's belief, and it is more undervalued than risk-free straight debt. In addition, the firm will also face a financial distress cost C^B if the cash flow at time 2 is X^L. However, we show in the proof of the above proposition that, while the firm can minimize the expected cost of financial distress by reducing the face value of debt using embedded equity options (i.e., convertible debt), risky straight debt will have the same undervaluation cost as convertible debt if the face value F of the convertible debt is greater than or equal to the cash flow X^L in the bad state. Even though the embedded equity component of convertible debt will be more undervalued than risky straight debt by the marginal investor, the straight debt component of the convertible debt will be less undervalued than risky straight debt, as the face value of convertible debt will be less than the face value of risky straight debt. Since these two effects cancel each other out, risky straight debt will have the same undervaluation cost as convertible debt for any face value of convertible debt where $F \geq X^L$. We also know from lemma 3 that the firm has no incentive to reduce the face value of the convertible debt strictly below X^L, since this increases the undervaluation of convertible debt relative to risky straight debt. Thus, the firm will optimally set the face value F of convertible debt to X^L to avoid costs of financial distress and any incremental undervaluation. Hence, the embedded option effect will dominate the undervaluation effect, and

the firm will prefer to issue convertible debt rather than risky straight debt when the required investment amount is large.[24]

We now relax the assumption that the issue cost C^I is always prohibitively large enough that the firm finds it optimal to issue only one type of security. We start with the simplest case in which the menu of securities available to the firm consists of equity, straight debt, or a combination of equity and straight debt. The following proposition characterizes the firm's choice of external financing between equity alone, straight debt alone, and a combination of straight debt and equity when the firm faces issue costs and costs of financial distress.

Proposition 6. (The choice between equity alone, straight debt alone, and a combination of straight debt and equity)
Let $\theta^f < \theta^m + d$.

1. The firm will choose to issue equity alone if outsiders are very optimistic about the firm on average, and their beliefs are very dispersed so that the marginal outside investor's belief $\hat{\theta}$ is above an upper threshold belief θ_{1b}, that is, $\hat{\theta} \geq \theta_{1b}$. The threshold belief θ_{1b} is less than the threshold θ_1 given in proposition 2, and decreasing in the issue cost C^I.

2. If the marginal outside investor's belief $\hat{\theta}$ is between a lower threshold belief θ_{2b} and the upper threshold belief θ_{1b} (i.e., $\theta_{2b} \leq \hat{\theta} < \theta_{1b}$), the firm will choose to issue either a combination of risk-free straight debt and equity if the issue cost is low $(C^I \leq \overline{C}_1^I)$ or straight debt alone if the issue cost is high $(C^I > \overline{C}_1^I)$.

3. If the marginal outside investor's belief $\hat{\theta}$ is below the lower threshold belief θ_{2b} (i.e., $\hat{\theta} < \theta_{2b}$), the firm will choose to issue either a combination of risky straight debt and equity (with the amount of equity issued smaller than in 2 above, with the face value of debt issued larger than in 2), if the issue cost is low $(C^I \leq \overline{C}_2^I)$ or straight debt alone if the issue cost is high $(C^I > \overline{C}_2^I)$. The threshold belief θ_{2b} is less than the threshold θ_2 given in proposition 2, and decreasing in the cost of financial distress C^B.

Similar to the intuition behind proposition 2, when the average outside investor is very optimistic about the firm's future cash flows and the dispersion in outsider beliefs is high, the equity price determined by the marginal outside investor will be much higher than the insiders equity valuation. In this case, the above proposition shows that the firm will issue equity alone to fully capture the greater optimism of outside

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[24] For a numerical example illustrating the optimal design of the convertible debt issued in this case, see Appendix A.

investors provided that the marginal outside investor's belief $\hat{\theta}$ exceeds a certain threshold value θ_{1b}. Conversely, when the average outside investor is less optimistic and the dispersion in outsider beliefs is lower, the marginal outside investor will value the firm only slightly higher or lower than will firm insiders. Then, the firm will choose between equity alone and a combination of debt and equity based on the following trade-off. On the one hand, issuing a combination of debt and equity allows the firm to raise some money by selling equity to the most optimistic investors and the rest by selling debt to less optimistic outsiders. In this way, the firm can capture the optimism of the very optimistic investors in the market by selling equity to them at a relatively high price. By issuing debt to less optimistic outside investors to raise the remaining amount, the firm will reduce its undervaluation costs, since the pricing of straight debt is much less sensitive to the beliefs of outside investors than the pricing of equity. On the other hand, issuing a combination of debt and equity also means that the firm has to pay issue costs on two tranches of securities instead of just one. Therefore, this proposition shows that the marginal outsider's threshold belief θ_{1b}, above which the firm prefers to issue equity alone in our full-fledged model, is less than the threshold θ_1 given in proposition 2 (in our basic model), and decreasing in the issue cost C^I.

When the firm prefers to issue a combination of debt and equity rather than equity alone (i.e., $\hat{\theta} < \theta_{1b}$), the marginal outside investor's belief $\hat{\theta}$ and the cost of financial distress C^B determine the proportion of debt versus equity that it chooses to issue. For a given level of required investment funding I, if the marginal outside investor is moderately optimistic (i.e., $\theta_{2b} \leq \hat{\theta} < \theta_{1b}$), the firm will choose to issue a combination of risk-free straight debt and equity. One should recall from proposition 2 that as the average optimism of outsiders and (or) the dispersion in their beliefs d decrease, the marginal outside investor becomes less optimistic, and therefore, the cost of issuing undervalued equity increases. As a result of this, the firm chooses to issue a larger amount of debt (I_D) in combination with equity ($I - I_D$). On the other hand, if the amount of debt that needs to be issued to meet the firm's investment requirement is sufficiently large ($I_D > X_L$), and therefore, the debt is risky, the firm will also face the risk of incurring a cost of financial distress C^B. In this case, if the cost of financial distress C^B is substantial and the marginal outside investor is not very pessimistic, the firm will tolerate issuing undervalued equity to a greater extent (compared to the situation in our basic model) and will choose to issue a combination of risk-free debt ($I_D \leq X_L$) and equity. However, if the marginal outside investor is rather more pessimistic about the firm's future cash-flow prospects (i.e., $\hat{\theta} < \theta_{2b}$) and the cost of financial distress C^B is not too large, the firm will choose to issue a combination of (a larger amount of) risky straight debt ($I_D > X_L$) and

equity, since the cost of issuing undervalued equity as a result of the marginal investor's pessimism will be greater than the expected cost of financial distress. This threshold level of marginal outsider's belief θ_{2b}, below which the firm prefers to issue a combination of risky debt and equity, is decreasing in the cost of financial distress C^B, and it is less than the similar threshold belief θ_2 in our basic model.

The above proposition also shows that, when the marginal outside investor is not very optimistic, the firm will choose to issue straight debt alone rather than a combination of debt and equity, when the issue cost C^I is very large. However, the firm will choose to issue a combination of debt and equity when the issue costs are relatively low compared to the price differentiation benefits of the debt-equity combination.

We now include convertible debt in the menu of securities available to the firm as well. Given that we have already analyzed the firm's choice between equity alone, debt alone, and convertible debt alone in the previous subsection, we will now confine ourselves to analyzing the optimality of the firm issuing convertible debt versus a combination of debt and equity.

Proposition 7. (The choice between convertible debt alone and a combination of straight debt and equity)
Suppose that outsiders are not very optimistic about the firm on average, and their beliefs are not very dispersed so that the marginal outside investor's belief $\hat{\theta}$ is below the upper threshold belief θ_{1b} given in proposition 6; that is, $\hat{\theta} < \theta_{1b}$. Then

1. If the marginal outside investor's belief $\hat{\theta}$ is between the lower threshold belief θ_{2b} and the upper threshold belief θ_{1b} (i.e., $\theta_{2b} \leq \hat{\theta} < \theta_{1b}$), the firm will choose to issue either a combination of risk-free straight debt and equity if the issue cost is low ($C^I \leq \overline{C}_3^I$) or convertible debt alone if the issue cost is high ($C^I > \overline{C}_3^I$).

2. If the marginal outside investor's belief $\hat{\theta}$ is below the lower threshold belief θ_{2b} (i.e., $\hat{\theta} < \theta_{2b}$), the firm will choose to issue either a combination of risky straight debt and equity (with the amount of equity issued smaller than in 1 above, with the face value of debt issued larger than in 1), if the issue cost is low ($C^I \leq \overline{C}_4^I$) or convertible debt alone if the issue cost is high ($C^I > \overline{C}_4^I$). The threshold belief θ_{2b} is decreasing in the firm's cost of financial distress C^B.

Issuing a combination of straight debt and equity means that the firm can sell its package of securities at higher prices because the firm can sell equity to the most optimistic outside investors at a relatively high price and sell debt to the less optimistic outsiders. In contrast, when the firm issues

convertible debt, the equity component and the debt component of the security have to be sold to the same group of investors, and therefore the firm is unable to capture the optimism among the more optimistic outside investors, since the price of the entire package of hybrid securities (i.e., convertible debt) will be determined by the belief of a single marginal investor (rather than by the beliefs of two different marginal investors, the beliefs of one for equity and one for debt). On the other hand, issuing convertible debt rather than a combination of debt and equity can save the firm issue costs, the benefit of which increases with the magnitude of the issue cost C^I. When the issue cost C^I is high, the cost saving benefit of convertible debt outweighs the valuation benefit of the debt-equity combination, and it is optimal for the firm to issue convertible debt. Conversely, when the the issue cost is low, the valuation benefit of the debt-equity combination outweighs the issue cost saving benefit of convertible debt, and it is optimal for the firm to issue a combination of debt and equity.

Finally, if the firm chooses to issue a combination of equity and debt, the fraction of equity versus debt issued in this combination depends on the belief of the marginal outside investor: if the marginal outsider's belief is as specified in 1 (i.e., $\theta_{2b} \leq \hat{\theta} < \theta_{1b}$), then the firm issues a significant amount of equity, so that the debt issued is risk-free; in contrast, if the marginal outsider's belief is lower, as specified in 2 (i.e., $\hat{\theta} < \theta_{2b}$), then the firm issues a small amount of equity and a larger amount of debt compared to the situation in 1, so that the debt issued is risky. As discussed under proposition 6, the threshold value of marginal outsider belief θ_{2b}, below which the firm issues a combination of risky debt and equity (rather than risk-free debt and a larger amount of equity), will be lower as the firm's cost of financial distress C^B is larger.[25]

3. The Price Impact of Security Issues

In this section, we investigate the price impact of equity, straight debt, and convertible debt issues on the current stock price of the firm at the time of a security issue. The price impact of a security issue is measured as the abnormal return to the firm's equity from the price prevailing before a security issue (not the announcement date) to the price prevailing after the issue.[26] Since the market already is aware that a security issue has been announced, one would expect a price impact of zero in the absence of

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[25] Since the threshold beliefs involved are the same across propositions 6 and 7, a combination of these propositions can be viewed as comparing all four financing choices available to the firm, namely, equity alone, debt alone, a combination of equity and debt, and convertible debt. We chose to split up these comparisons across two propositions mainly for clarity of exposition.

[26] Note that, empirically, the price impact of a security issue is quite different from an announcement effect in the abnormal return measured on the day of the announcement of the security issue (before the issue becomes effective), while the price impact is the abnormal return measured on the day the security issue actually comes into effect.

heterogeneity in investor beliefs. While, in practice, some time will elapse between the announcement date and the issue date of a security, for simplicity, we model the issue date and the announcement date together.[27]

Proposition 8. (Price impact of security issues)

1. If the firm issues equity at time 0, there is a negative impact on the stock price on the issue date:

$$\Delta PE^{Equity} = -\left(1 - \frac{I}{\hat{\theta}X^H + (1-\hat{\theta})X^L}\right)\frac{2dI}{W}(X^H - X^L) < 0. \quad (16)$$

2. If the firm issues straight debt or convertible debt, there is no impact on the stock price on the issue date:

$$\Delta PE^{Debt} = \Delta PE^{Convertible} = 0. \quad (17)$$

3. The greater the dispersion in outsiders' beliefs, d, the greater the price impact of an equity issue

$$\frac{\partial|\Delta PE^{Equity}|}{\partial d} > 0. \quad (18)$$

When the firm issues equity, it must sell the equity to investors who are less optimistic about the firm's value than current shareholders (since the current shareholders have limited wealth). Hence, the valuation of the new marginal equity investor (i.e., the stock price just after the equity issue) will be lower than the valuation of the marginal equity investor before the equity issue. This results in a fall in the firm's share price, yielding a negative price impact. On the other hand, if straight debt or convertible debt is issued, the equity price will remain at the same level as before the security issue. This is because the marginal investor who holds equity in the firm remains the same in these cases, so that the valuation of the marginal equity holder is unaffected, resulting in a zero price impact of the straight debt issue or convertible debt issue on the firm's equity.[28],[29] This proposition also shows that the negative price impact

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[27] If we separate the two dates, there should be no announcement effect in our setting since investors do not update their beliefs based on others' actions (insiders actions do not convey any information to outsiders in our setting since there is no information asymmetry in our model).

[28] However, if the firm has outstanding debt, the debt issue will have a negative impact on the price of the firm's debt, through a mechanism similar to that generating a negative price impact of an equity issue on the firm's outstanding equity.

[29] Note that the mechanism generating a differential price impact of an equity issue versus a debt issue on a firm's outstanding equity in our setting of heterogeneous beliefs is completely different from that

of an equity issue will be larger in absolute value as the dispersion in outsiders' beliefs increases. If outsiders' beliefs are more dispersed (d is greater), the distance the firm has to go down the ladder of outside investors' beliefs (to raise the entire investment amount of I) increases, yielding a more negative price impact of an equity issue.[30]

4. Long-Run Stock Returns following Security Issues

In this section, we will analyze the long-run stock returns of firms following equity, straight debt, and convertible debt issues. Here, we extend our basic model presented in Section 1 in a new direction by allowing all agents to engage in Bayesian learning based on additional noisy public information about the firm's prospects arriving subsequent to the firm's security issue. To model the arrival of this additional noisy information, we introduce another date (time 2) between the security issue date (time 1) and the final date (which we now denote as time 3), when noisy new information about the firm's prospects becomes available to outsiders. The sequence of events in this modified model is given in Figure 2.[31]

This noisy new information is hard and credible: an example is the information that can be collected from the firm's quarterly reports and earnings announcements. We model the arrival of new public information at time 2 and the revision in outsider beliefs in response to this information as follows. Let T be the true probability (unobservable to all agents) of the firm's project realizing the high cash flow X^H with the low cash flow X^L realized with probability $(1 - T)$. At time 2, all agents observe the noisy public signal t. Each agent believes that the signal gives them perfect information about the high cash-flow probability T with probability δ (thus each agent believes that in this case $t = T$), and that

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generating differences in announcement effect of equity and debt issues on the firm's equity in asymmetric information models, such as Myers and Majluf (1984).

[30] Chemmanur et al. (2011) show that there is indeed such a negative price impact on the date of an equity issue, and this negative impact is increasing in the dispersion in outsider beliefs. One may wonder why such a price impact is not arbitraged away by traders who short sell the firm's equity on the announcement date of the equity issue (and buy it back after the share price has fallen on the issue date), thus moving forward the price drop to the announcement date. Note that such a trade is not a riskless arbitrage, since there will be market movements in the weeks between the announcement date and the issue date. As Mitchell, Pulvino, and Stafford (2002) show in the analogous context of risky arbitrage around negative stub values in equity carveouts, traders attempting such arbitrage often earn a rate of return lower than the risk-free interest rate.

[31] Clearly, an important development that occurs in the months subsequent to any security issue is the arrival of additional information about the firm's postissue operating performance, and it is rational to incorporate the reaction of investors to such information in any analysis of long-run stock returns. While, for simplicity of analysis, we have not incorporated such Bayesian learning into our basic model of Section 1 and our full-fledged model of Section 2, it should be noted that all our results of Sections 1 and 2 go through qualitatively unchanged even in the presence of such Bayesian learning that allows the revision of agents' heterogeneous priors based on additional noisy information.
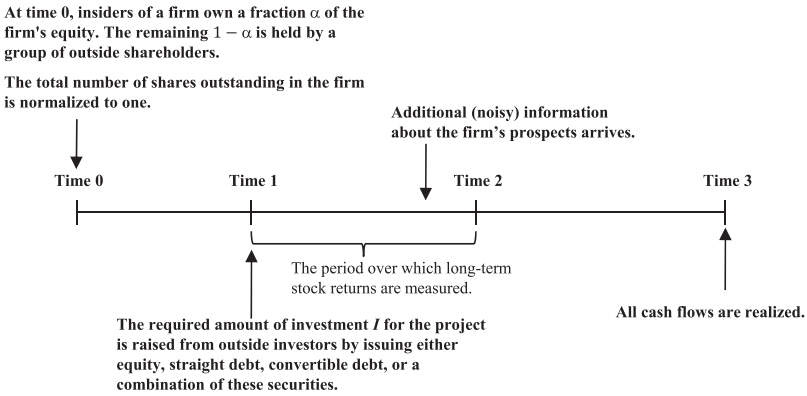
At time 0, insiders of a firm own a fraction α of the firm's equity. The remaining $1 - \alpha$ is held by a group of outside shareholders.

The total number of shares outstanding in the firm is normalized to one.

Additional (noisy) information about the firm's prospects arrives.

| Time 0 | Time 1 | Time 2 | Time 3 |

The period over which long-term stock returns are measured.

All cash flows are realized.

The required amount of investment I for the project is raised from outside investors by issuing either equity, straight debt, convertible debt, or a combination of these securities.

Figure 2
Sequence of events in the extended model

it is completely uninformative about T with the remaining probability (so that they stick to their prior belief θ about the probability of a high cash flow realization). After observing t at time 2, each agent updates his prior belief θ about the firm's cash flow to the posterior belief given by $\delta t + (1 - \delta)\theta$. Therefore, outsiders' posterior will range from $\delta t + (1 - \delta)(\theta^m - d)$ to $\delta t + (1 - \delta)(\theta^m + d)$, with a dispersion of $(1 - \delta)d$. Note that the reduction δd in the dispersion in outsider beliefs is proportional to the initial dispersion d. Thus, due to the arrival of this new public information, outside investor beliefs about the firm's cash flows become less heterogeneous in the long run. In particular, we show that, in the long run (i.e., at time 2), the dispersion in outsider beliefs decreases from d to $(1 - \delta)d$, where δ is the percentage reduction in the dispersion in outside investors' beliefs about the firm's future cash flow.

Clearly, the reduction δd in the dispersion in outsider beliefs is increasing in the precision δ of the new information available to outsiders. It is useful to compare two extreme realizations of δ: when $\delta = 0$, there will be no reduction in the dispersion of outsider beliefs, since, in this case, the signal is totally uninformative; when $\delta = 1$, every outside investor will have the same common posterior belief $t = T$ as the signal contains no noise.

Note that the posterior mean belief of investors at time 2 is equal to $\delta t + (1 - \delta)\theta^m$, a weighted average of the new signal t and the prior mean belief θ^m. For simplicity, we assume that the signal t is equally likely to be above or below θ^m (i.e., equally likely to be either $\theta^m + \eta$ or $\theta^m - \eta$, where $\eta > 0$), so that the time-1-expected value of investors' posterior (time 2) mean belief is also equal to the prior mean belief θ^m.[32]

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[32] This is also the natural assumption to make in the absence of any systematic bias (either upward or downward) at time 1 about the future cash-flow performance of the firm.

We now analyze the long-run stock returns of firms issuing equity in the above setting. For a firm that issues equity alone at time 1 to raise the investment amount I, the marginal equity investor's belief at time 1 is equal to $\hat{\theta} = \theta^m + d\left(1 - \frac{2I}{W}\right)$. If the dispersion in investor beliefs about the firm goes down by an amount of δd by time 2, the time-1-expected value of the marginal equity investor's belief at time 2 will be equal to $\hat{\theta}_2 = \theta^m + d(1 - \delta)\left(1 - \frac{2I}{W}\right)$, and the time 1 expectation (based on the marginal investor's belief) of the time 2 market value of the firm will be given by:

$$E[V_2] = \hat{\theta}_2 X^H + (1 - \hat{\theta}_2)X^L. \tag{19}$$

Proposition 9. (Long-run stock returns following equity issues)
The long-run stock return of a firm issuing equity is expected to be negative and is given by

$$LR^{Equity} = \frac{E\left[PE_2^{Equity}\right] - PE_1^{Equity}}{PE_1^{Equity}} = \frac{-d\delta(X^H - X^L)\left(1 - \frac{2I}{W}\right)}{\hat{\theta}X^H + (1 - \hat{\theta})X^L} < 0. \tag{20}$$

The above proposition states that the long-run stock return of firms following equity issues is always expected to be negative. The key to understanding the intuition here is to recall that the firm's equity at time 1 is priced by the marginal outside investor whose belief $\hat{\theta}$ is determined by going down the ladder of investor beliefs (starting with the most optimistic outsider belief $(\theta^m + d)$) until the entire amount of I is raised. Thus, the higher the dispersion in outsider beliefs d at the time of security issue (time 1), the higher the marginal equity investor's belief $\hat{\theta}$ at time 1, and therefore, the higher the firm's share price at the time of the equity issue. As additional noisy public information about the firm's future cash flows arrives and all outsiders revise their heterogeneous prior beliefs based on this noisy information, all investors with beliefs above θ^m are expected to become less optimistic about the firm. This implies that the existing marginal equity investor in the firm is also expected to become less optimistic about the firm, so that the firm's stock price is expected to fall in the long run (at time 2) correspondingly. Thus, the average long-run stock returns of firms subsequent to equity issues will always be negative.[33]

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[33] Similarly, all investors with beliefs below θ^m at the time of security issue will become more optimistic as the spread around the mean belief level decreases over time. However, since the prior belief of the marginal investor is above θ^m, the beliefs of the latter group of investors do not affect long-run stock returns.

Proposition 10. (Comparative statics on long-run stock returns following equity issues)

The average long-run stock return subsequent to an equity issue is decreasing in the initial dispersion in outsiders' beliefs d at the time of equity issue and the informativeness δ of the noisy public information arriving after the equity issue; that is, $\frac{\partial LR^{Equity}}{\partial d} < 0$ and $\frac{\partial LR^{Equity}}{\partial \delta} < 0$.

When the hard information about the firm that arrives after an equity issue is more informative (less noisy), the reduction in the dispersion in outsiders' beliefs is greater since investors update their beliefs about the firm using this new public information. Consequently, the marginal outside investor's optimism is greater, resulting in more negative, on average, long-run stock returns following equity issues. Since the long-run reduction in the dispersion in outside investor beliefs is given by δd, the long-run stock returns of equity issues will be decreasing in both the initial dispersion d and the informativeness δ of the additional information arriving in the long run after the equity issue.

Proposition 11. (Long-run stock returns following straight debt and convertible debt issues)

1. The long-run stock return of a firm issuing straight debt or convertible debt is expected to be negative and is given by

$$LR^{Debt} = LR^{Convertible} = \frac{-d\delta}{\theta^m + d} < 0. \qquad (21)$$

2. The long-run stock returns of firms issuing straight debt or convertible debt are decreasing in the initial dispersion in outsiders' beliefs d at the time of equity issue and the informativeness δ of the noisy public information arriving after the equity issue; that is, $\frac{\partial LR^{Debt}}{\partial d} < 0$ and $\frac{\partial LR^{Debt}}{\partial \delta} < 0$.

The intuition behind the above proposition is as follows. Unlike in the case of an equity issue, issuing straight debt or convertible debt does not affect the identity of the marginal investor in the firm's equity. Therefore, the stock price of the firm immediately after a debt (either straight or convertible) issue will reflect only the beliefs of the current marginal investor in the firm's equity, since the marginal investor in the firm's equity prior to the debt issue will be the same as the marginal investor in the firm's equity after the debt issue.[34] Further, the long-run (time 2) stock

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[34] In other words, the relevant marginal investor in determining the long-run stock returns following a straight debt or a convertible debt issue is the marginal investor in the firm's equity at the time of the debt issue (since the marginal investor in the firm's equity does not change as a result of the debt issue). In contrast, the marginal investor determining firms' long-run stock returns following an equity issue is the

price of the firm will reflect the belief of the same marginal equity investor who is expected to have an updated belief of $\theta^m + d(1 - \delta)$ at time 2, as discussed earlier. Therefore, since the marginal equity investor becomes less optimistic in the long run after all outside investors revise their heterogeneous prior beliefs based on noisy additional information, the long-run stock return following a debt (or a convertible debt) issue is also expected to be negative. Further, similar to the case for an equity issue, the average long-run stock return following a debt issue will also be decreasing in the initial dispersion in outside investors' beliefs and the informativeness of the additional information arriving after the debt issue.

We now compare the average long-run stock returns of firms following equity, straight debt, and convertible debt issues. Given insiders' beliefs and market conditions (i.e., the mean belief of outsiders and the dispersion in outsider beliefs), any given firm will issue only one kind of security in equilibrium, namely, equity, straight debt, or convertible debt: in other words, it does not make sense to compare the long-run stock returns of a given firm facing identical market conditions issuing equity versus issuing debt, since only one will occur in equilibrium.[35] Therefore, we can only make comparisons across the average long-run stock returns of samples of firms issuing each security. The following proposition compares the average long-run stock return of firms following issues of equity, straight debt, and convertible debt.

Proposition 12. (Comparison of average long-run stock returns across security issues)
Let $\theta^f > \theta^m$, and let (B119) hold. Then, the average long-run stock return of firms following a debt issue (straight debt or convertible debt) will be less negative than that following an equity issue; that is, $\overline{LR}^{Debt} = \overline{LR}^{Convertible} > \overline{LR}^{Equity}$.

We showed in propositions 1 and 5 that a firm prefers to issue equity alone rather than debt alone if the marginal outside investor is more optimistic than firm insiders. We also showed that, if the marginal outside investor is more pessimistic than firm insiders, the firm issues convertible

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marginal investor holding the firm's equity immediately after the equity issue (since the marginal investor in the firm's equity after the equity issue is different from the marginal investor in the equity prior to the equity issue).

[35] While, in Sections 1 and 2, we show that firms may also choose to issue combinations of equity and debt under certain circumstances, in this section we confine ourselves to comparing the long-run stock returns of firms issuing equity alone, straight debt alone, or convertible debt alone since the empirical literature has focused on only these three securities. However, the general intuition behind our results here go through even if we were to consider firms issuing a combination of equity and debt, provided we classify firms raising a larger fraction of the total amount of external financing by issuing equity as equity issuers, and those raising a larger fraction of the external financing amount by issuing debt as debt issuers.

debt or straight debt depending on the size of required investment I and the firm's cost of financial distress. Recall that the marginal outside investor is more optimistic than firm insiders only if the mean outsider belief, and the dispersion in outside investors' beliefs are sufficiently high. Thus, for a given level of mean outside investor beliefs, we know that the dispersion in outsiders' beliefs at the time of the security issue will be higher for equity-issuing firms than that for debt-issuing firms. Consequently, the expected fall in the marginal investor's belief as additional information (of given precision) becomes publicly available will be greater in the case of firms issuing equity compared to those issuing debt. Therefore, the average long-run stock return of firms that issue (straight or convertible) debt will be less negative than the average long-run stock return of those that issue equity.

5. Empirical Implications

We now highlight some of the testable implications of the model and their relationship to the existing empirical literature.

1. The pecking order of external financing under heterogeneous beliefs: Our model implies a "pecking order" of external financing under heterogeneous beliefs different from that implied by asymmetric information models. If the marginal outside investor is much more optimistic than firm insiders, the firm issues the most belief-sensitive security (equity only) to capture outsiders' optimism, rather than straight debt or convertible debt, which are less belief-sensitive securities, and therefore less overvalued than equity. If outside investors are moderately more optimistic or less optimistic than firm insiders on average, and the dispersion in their beliefs is not too large, the firm will issue a combination of equity and debt to cater to outside investor clienteles with different beliefs about the firm's future prospects. In particular, if the marginal outsider's belief is above a certain threshold belief, then the firm issues a combination of a significant amount of equity and some risk-free debt. If, however, the marginal outsider's belief is below this threshold value, the firm issues a small amount of equity in combination with a significant amount of risky straight debt to raise the required investment amount.[36] It is important to note that the above pecking order of external financing is quite different in a setting where heterogeneity in beliefs is the sole market imperfection

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[36] Thus, our predictions can explain some of the survey evidence of Graham and Harvey (2001). They find that two-thirds of CFOs agree that "the amount by which our stock is undervalued or overvalued was an important or very important consideration" in issuing equity, and nearly as many agree that "if our stock price has recently risen, the price at which we can sell is high." In that survey, equity market prices are regarded as more important than 9 out of 10 other factors considered in the decision to issue common stock. Asymmetric information models of equity issues with rational investors cannot explain such findings, since rational investors would appropriately discount the valuations of firms issuing equity in a setting of asymmetric information, as demonstrated by Myers and Majluf (1984).

compared to one where asymmetric information is the sole market imperfection. First, while issuing equity is the last choice in an asymmetric information setting (e.g., Myers and Majluf 1984), it is the first choice in a setting where outside investors are optimistic enough so that the marginal outside investor's belief is above that of firm insiders.[37] Second, under asymmetric information, if the firm can raise the required amount of external financing by issuing risk-free debt, this will be the most preferred security to issue; in contrast, even when the marginal outside investor is pessimistic relative to firm insiders, under heterogeneous beliefs, the firm prefers to issue a combination of equity and debt rather than risk-free debt alone to raise the required external financing. Third, under asymmetric information, if the firm cannot raise the entire amount of financing required by issuing risk-free debt, it will choose to issue risky debt (or other securities that are less information-sensitive than equity) to raise this amount; in contrast, under heterogeneous beliefs, the firm will raise the required amount by issuing a combination of equity and risky debt under these circumstances, even when the marginal outside investor is more pessimistic than firm insiders.

2. Relationship between investor optimism, dispersion in investor beliefs, and the choice of equity versus debt: Our model predicts that the greater the level of optimism (average belief) among outsiders, and the greater the dispersion in outsider beliefs (or both), the more likely the firm is to choose to issue equity rather than debt. This is because the belief of the marginal investor in the firm's equity is more likely to be above that of firm insiders if the level of optimism, dispersion, or both among outsiders is higher. Evidence supporting this prediction is provided by Chemmanur et al. (2011). This prediction also provides a fully rational explanation of the empirical results of Baker and Wurgler (2002), who document that firms tend to issue equity (rather than debt) when outsider optimism is greatest.

3. Relationship between investment amount and the choice of equity versus debt: Our model predicts that, the greater the investment amount to be raised by the firm, the less likely it is to issue equity rather than debt. This is because, since each investor has limited wealth to invest in the firm, the beliefs of the marginal investor is more likely to fall below that of insiders as the amount raised by the firm is greater. Evidence supporting this prediction is provided by Chemmanur et al. (2011), who document that the greater the investment amount to be raised by the firm, the less likely it is to issue equity rather than debt.

4. Sequential equity or debt offerings and the "private placement discount": Our analysis of sequential tranching in Section 1.3 gives a new

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[37] The literature on security issuance and security design under asymmetric information comprises a number of other important papers (see, e.g., Giammarino and Lewis 1988; Nachman and Noe 1994).

rationale for firms making sequential offerings of equity or debt within a short period of time (in the absence of any new information between these sequential offerings). It also predicts that, in the case of a tranched equity (or debt) issue, the valuation of the subsequent equity (or debt) offerings will be lower than that of the first offering in the sequence. Our model also gives a new rationale for the well-documented "private placement discount" (see, e.g., Hertzel et al. 2002; Wu 2004), where a firm makes an equity offering to certain investors in a private equity placement at a price significantly below the prevailing stock price. In our setting, a private equity placement will be undertaken by a firm which wishes to make a second public equity issue, but realizes that there may not be a large enough pool of investors who are sufficiently optimistic about the firm to justify the relatively high issue costs (e.g., investment banking fees) involved. A private placement of equity at a discount to a prior public equity issue may be a solution to this problem, since it allows the firm to raise equity at a low transaction cost without lowering its publicly traded stock price (since the belief of the marginal investor holding the firm's publicly traded equity does not go down in this case).[38]

5. *A new rationale for issuing convertible debt:* Our model suggests a new rationale for issuing convertible debt.[39] When a firm faces significant issue costs (so that it prefers to raise the required external financing by issuing a single security) and the marginal outside investor in an external financing is likely to be less optimistic about the firm's prospects compared to firm insiders (so that issuing equity alone is suboptimal), then the firm will issue convertible debt in equilibrium, provided that the costs of financial distress in its industry are also nontrivial. Recall that, when financial distress costs are significant, issuing convertible debt dominates issuing risky straight debt alone, since, given that the option to convert to equity embedded in a convertible has a positive value, the face value required to be offered to bond holders is always lower for a convertible debt issue than for a straight debt issue, thus leading to smaller expected financial distress costs for a convertible debt issue. In other words, convertible debt is a mechanism for minimizing the aggregate issue costs and

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[38] Thus, a private placement of equity can be viewed as the second tranche of a sequential equity offering in the setting of our sequential tranching analysis of section 1.3.

[39] Note that this rationale for issuing convertibles based on heterogeneous beliefs is completely unrelated to rationales given in the existing literature based on asymmetric information (see, e.g., Brennan and Kraus 1987; Constantinides and Grundy 1989; Stein 1992). In particular, in Stein's (1992) model, there are three types of firms: good, medium, and bad. The higher types incur a lower probability of realizing the low cash flow and being in financial distress (thus incurring the distress cost) consequently. The equilibrium in that model is a fully separating (signaling) equilibrium, with the good type issuing straight debt, the medium type issuing convertible debt, and the bad type issuing equity, with the cost of financial distress serving as a signaling cost that allows the good firm type to distinguish itself from the medium type and the medium type to distinguish itself from the bad type. Our rationale for issuing convertibles is also different from that developed by Green (1984), who models convertibles as a means of mitigating the risk-shifting (agency) problem in a setting of conflicts of interest between stockholders and bondholders.

the expected financial distress costs incurred by a firm, when issuing equity alone is suboptimal and the amount of external financing is large enough that this financing requirement cannot be met by issuing risk-free debt alone. Consistent with this, there is significant evidence indicating that convertible debt is issued by smaller, riskier, and high-growth firms in industries characterized by higher costs in the event of financial distress, for example, high-tech firms.

6. The price impact of an equity issue: The price impact refers to the return to a firm's equity upon the actual issue (not the announcement) of a security (in our context, equity, straight debt, or convertible debt). The price impact will be given by the change in share price of the firm's equity (or return) from the price prevailing before the security issue to the price prevailing after the issue. In our setting, the price prevailing before an equity issue will be determined by the beliefs of the marginal investor holding the firm's equity prior to the issue. Since current shareholders have limited wealth, when new equity is issued it will have to be sold to new investors who are less optimistic about the firm's long-run prospects, so that the beliefs of the marginal investor after the equity issue will be less optimistic compared to that before the equity issue, resulting in a lower share price after the equity issue: that is, the price impact of an equity issue will be negative. Further, the above fall in share price upon an equity issue will be greater as the dispersion in outsider beliefs is greater: that is, when dispersion is greater, the price impact will be more negative. To the best of our knowledge, this prediction is unique to our model. Evidence supporting these predictions is provided by Chemmanur et al. (2011), who find that the price impact of an equity issue is negative, and is decreasing (more negative) as the dispersion in outsider beliefs is greater.

7. The price impact of a debt or convertible debt issue: Our analysis predicts that the price impact of a straight debt or a convertible debt issue will be zero (and therefore smaller in magnitude than that of an equity issue). This is because no new equity is issued, so that the beliefs of the marginal equity investor before the firm's debt issue and that of the marginal equity investor after the debt issue will be the same in this case. In other words, the price of the firm's equity will be unchanged on the day of the debt issue, so that the price impact (on equity) of a straight or convertible debt issue will be zero. Consistent with this, the empirical analysis of Chemmanur et al. (2011) shows that the price impact of a straight debt issue is insignificant.

8. Long-run stock returns following an equity issue: We define long-run stock return as the one-, two-, or three-year stock return starting from the closing price on the day of an actual equity issue (as is standard in the relevant empirical literature). In other words, long-run stock returns exclude the price impact of a security issue. Our model predicts that the

long-run stock returns following an equity issue will be negative. Further, the greater the dispersion in outsider beliefs regarding a firm's prospects at the time of the equity issue, the more negative its long-run stock return following the equity issue will be. Evidence supporting this prediction is provided by Chemmanur et al. (2011), who document that the more optimistic outsiders are on average about a firms prospects at the time of an equity issue, and the more dispersed their beliefs, the more negative the long-run stock returns of the issuing firm.

9. *Long-run stock returns following a debt or a convertible debt issue:* Our model predicts that the average long-run stock return of firms following a (straight or convertible) debt issue will be negative, but greater (less negative) than the average long-run stock return of firms following an equity issue. Evidence supporting this prediction is provided by Chemmanur et al. (2011), who document that, while the average long-run stock returns to both equity and debt issuers are negative, the average stock return to debt issuers is significantly less negative than that of equity issuers (see also Spiess and Affleck-Graves 1995; Spiess and Affleck-Graves 1999).

6. Relation to the Theoretical Literature on Heterogeneous Beliefs and Disagreement

Our paper is related to four strands in the theoretical finance and economics literature on heterogeneous beliefs and disagreement.[40] The first is the literature on investor behavior and trading under heterogeneous beliefs. Duffie, Gârleanu, and Pedersen (2002) show that, even when short selling is allowed (but requires searching for security lenders and bargaining over the lending fee), the price of a security will be elevated and can be expected to decline subsequently in an environment of heterogeneous beliefs among investors if lendable securities are difficult to locate. Harris and Raviv (1993) use a model involving differences of opinion among traders about the interpretation of public information they receive to develop predictions regarding the relationship between stock price and volume. Several other authors have also examined the asset pricing and trading implications of heterogeneous beliefs (see, e.g., Harrison and Kreps 1978, Varian 1985, 1989, Kandel and Pearson 1995, Morris 1996, Duffie, Gârleanu, and Pedersen 2002, Chen, Hong, and Stein 2002 for contributions to this literature and Scheinkman and Xiong 2004 for a review).[41]

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[40] Due to space limitations, we will not review the large theoretical literature on capital structure driven by considerations other than heterogeneous beliefs or disagreement here. Please see Harris and Review (1991) for an excellent early review of this literature.

[41] There is also an empirical asset pricing literature based on heterogeneous beliefs (see, e.g., Diether, Malloy, and Scherbina 2002).

The second literature our paper is related to is the emerging literature on the corporate finance implications of heterogeneous beliefs. Allen and Gale (1999) examine how heterogeneous priors among investors affect the source of financing (banks versus equity) of new projects.[42] In contrast to their paper, our primary focus is on how heterogeneity in beliefs among investors affects the firm's choice of security to issue. Garmaise (2001) analyzes the optimal design of securities by a cash-constrained firm facing investors with diverse beliefs: however, his focus is on comparing optimal designs when investors have rational beliefs (in the sense of Kurz 1994) versus rational expectations.

The third literature our paper is related to is the one dealing with the foundations of heterogeneous beliefs models. Several authors have argued that prior beliefs should be viewed as primitives in the economic environment (Kreps 1990) and that it may be appropriate for economists to allow for differences in prior beliefs to understand economic phenomena (Morris 1995; Allen and Morris 1998). Morris (1995) provides a detailed discussion of the role of the common prior assumption in economic theory. Kurz (1994) provides the foundations for heterogeneous but rational priors.

The fourth and final literature our paper is related to is the corporate finance literature making use of the assumption of "disagreement" between firm insiders and outsiders. A prominent paper in this literature is Dittmar and Thakor (2007), who study a firm's choice between issuing debt and equity when insiders and outsiders disagree about the firm's choice of project to invest in. They assume that while equity holders disagree with insiders about project choice only based on their beliefs, debt holders may disagree with them for additional reasons (such as having a different objective function). The choice between equity and debt in their setting trades off the additional autonomy provided by equity holders to the manager in choosing projects against the tax benefits of debt: their model predicts that equity will be issued when there is less disagreement between insiders and outsiders and debt will be issued when this disagreement is more. Apart from the fact that there is no heterogeneity among outside investors' beliefs in their setting (unlike our setting where such heterogeneity is the driving force), the trade-off driving the debt versus equity choice in their model is quite different from ours (as discussed above). Further, their prediction regarding the conditions under which a firm will issue equity rather than debt is exactly opposite to that emerging from our analysis (in the sense that, while their model predicts that firms are more likely to issue debt when there is more disagreement between

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[42] See also Abel and Mailath (1994) who demonstrate that in certain special settings with heterogeneous beliefs, even projects that all investors believe have negative expected value if undertaken may be financed by these investors.

firm insiders and outsiders, our model predicts that firms are more likely to issue equity when there is greater heterogeneity in beliefs among outside investors, and the average outsider is more optimistic about the firm's future prospects compared to firm insiders).[43,44]

7. Conclusion

In this paper, we analyzed a firm's choice of capital structure, and the price impact and long-run stock returns following security issuance, in an environment of heterogeneous beliefs and short sales constraints. We studied a setting in which the insiders of a firm, owning a certain fraction of its equity, choose between equity, debt, or convertible debt to raise additional financing to implement a positive-NPV project. The insiders' objective is to maximize their long-run wealth conditional on their own beliefs about their firm's future prospects. Market participants, each of whom have limited wealth, have heterogeneous beliefs about the firm's long-run value. We analyzed two different economic settings: one in which there are no market imperfections other than heterogeneous beliefs, and another in which there are also significant costs of issuing securities and of financial distress. We showed that, in the absence of these two costs, the average belief of outsiders ("optimism") and the dispersion in outsider beliefs are the crucial determinants of the firm's security choice. When outsider beliefs are highly optimistic relative to that of firm insiders and the dispersion in outsider beliefs is high, the firm issues equity alone; when outsider beliefs are less optimistic (and less dispersed), the firm issues a combination of equity and debt. Neither straight debt alone nor convertible debt alone is optimal in this setting. Once the two costs are significant, we showed that the firm always issues equity when outsider beliefs are optimistic and highly dispersed. If outsider beliefs are less optimistic, the firm issues a combination of equity and debt if issue costs are small; if issue costs are large, it either issues risk-free straight debt (if the investment amount required is small) or convertible debt (if this amount is large). Our model generates a pecking order of external financing different from asymmetric information models and new predictions for capital structure, the price impact of security issues, and long-run stock returns following these issues. It also provides a new rationale for issuing convertible debt and the sequential tranching of securities.

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[43] Evidence in support of this prediction is provided by the empirical analysis of Chemmanur et al. (2011).

[44] See also Bayar, Chemmanur, and Liu (2011), who develop a model of equity carveouts and negative stub values in a setting of heterogeneous beliefs. Bayar, Chemmanur, and Liu (2014) develop a theory of a firm's choice between dividends and stock repurchases in a setting of heterogeneous beliefs, and for the long-run stock returns following dividend payments and stock repurchases in such a setting.

Appendix A: A Numerical Example of the Optimal Design of a Convertible Debt Issue

Consider the following illustration of the optimal design of a convertible debt issue in our full-fledged model of Section 2. Let $X^H = 100$, $X^L = 10$, $I = 50$, $W = 1$, 500, $\theta^m = 0.5$, and $d = 0.2$. Then, the marginal investor's belief is equal to $\hat{\theta} = 0.6867$. Suppose also that the insider belief is equal to $\theta^f = 0.8$. Each unit of convertible debt has a face value of one.

If financial distress costs are nonzero, proposition 5 implies that it is optimal for the firm to issue ten units of risk-free convertible debt with a total face value of $F = 10$. The pricing of convertible debt and investors' breakeven condition determine the conversion ratio x and the price p per unit of convertible debt. The conversion price per unit of convertible debt is $1/x$. $I = 50$ and $F = 10$ imply that $p = 5$, since $p = I/F$ (note that F also represents the number of units of convertible debt issued, since each unit has face value of 1). The pricing equation (B21) of convertible debt implies that $x = 0.214985$, and $1/x = 4.6515$. Since there is one share of equity outstanding initially, there will be $(1 + x F)$ shares of equity outstanding if convertible debt holders convert. Given that $1 + xF = 3.14985$, if convertible debt holders convert, there will be 3.14985 shares of equity outstanding.

Prior to conversion, we assume that the following condition holds: $X^L < F + 1/x < X^H$. This condition implies that the highest possible total face value of convertible debt is equal to the total face value of straight risky debt, which in this example is equal to $68.2524 > I = 50$. The corresponding conversion price is equal to 31.7476. Above this conversion price, convertible debt is equivalent to straight risky debt. This restriction also implies that the lowest possible value of F is 6.96, and the lowest possible value of the conversion price is 3.03621, since, below these values, convertible debt is equivalent to equity.

Convertible debt holders will convert to equity if and only if the cash flow per share of equity after conversion is greater than the conversion price $(1/x)$, which is 4.6515 here. If the low cash flow $(X^L = 10)$ is realized, the cash flow per share of equity is $10/3.14985 = 3.17476$. Since 3.17476 is less than the conversion price 4.6515, convertible debt holders do not convert. If the high cash flow $(X^H = 100)$ is realized, the cash flow per share of equity is $100/3.14985 = 31.7476$, and convertible debt holders convert.

In our example, if F is less than $X^L = 10$ (i.e., the conversion price is less than 4.6515), the undervaluation cost of issuing convertible debt will be higher than that of issuing risky straight debt. If $10 \leq F < 68.2524$, the undervaluation cost of issuing convertible debt will be the same as that of issuing risky straight debt. However, if the firm sets F strictly above ten (between 10 and 68.2524), it will also incur a positive financial distress cost. Thus, it is optimal for the firm to set the face value of the convertible, $F = 10$.

Appendix B: Proofs of Lemmas and Propositions

Proof of Lemma 1

If the equity is traded at a price of PE_1^{Equity} per share at time 1, when the firm issues equity, all investors whose valuation higher than PE_1^{Equity} will be willing to buy. Denote $\hat{\theta}$ as the belief of the marginal investor, whose valuation of the equity equals the market price PE_1^{Equity}. Because potential outside investors have a total wealth of W, and they are uniformly distributed over the interval with a length of $2d$, each investor has a wealth of $\frac{W}{2d}$. Because the firm needs to raise an amount I from investors in the interval $[\hat{\theta}, \theta^m + d]$, we have

$$\int_{\hat{\theta}}^{\theta^m + d} \frac{W}{2d} d\theta = I. \tag{B1}$$

The above equation means that the total wealth of those who buy the new issues equals the amount the firm wants to raise, I. Solving for $\hat{\theta}$, we have

$$\hat{\theta} = \theta^m + d - \frac{2dI}{W}. \tag{B2}$$

The market price of new shares sold should be determined by the marginal investor's valuation of the shares, that is,

$$PE_1^{Equity} = \frac{[\hat{\theta} X^H + (1 - \hat{\theta}) X^L]}{1 + E_1}, \tag{B3}$$

where the left side is the market price of each share of equity and the right side is the marginal investor's valuation of each share of equity. Further, the amount raised by the firm is I, which means

$$PE_1^{Equity} \times E_1 = I. \tag{B4}$$

Solving for Equations (B3) and (B4) leads to

$$PE_1^{Equity} = \hat{\theta} X^H + (1 - \hat{\theta}) X^L - I, \tag{B5}$$

and

$$E_1 = \frac{I}{\hat{\theta} X^H + (1 - \hat{\theta}) X^L - I}. \tag{B6}$$

The expected payoff to the firm's current shareholders is

$$EU^{Equity} = \left(1 - \frac{I}{\hat{\theta} X^H + (1 - \hat{\theta}) X^L}\right)[\theta^f X^H + (1 - \theta^f) X^L]. \tag{B7}$$

Q.E.D.

Proof of Lemma 2

Suppose the firm needs to issue F units of straight debt to raise the amount I. Because potential outside investors have a total wealth of W, and they are uniformly distributed over the interval with length of $2d$, the marginal investor in the firm's debt is also $\hat{\theta} = \theta^m + d\left(1 - \frac{2I}{W}\right)$, similar to the argument in the proof of lemma 1.

First, assume that $X^H > I > X^L$, that is, the debt is risky. The payoff to each unit of straight debt is one in the good state and $\frac{X^L}{F}$ in the bad state, so the market price of each unit of debt, which is determined by the marginal investor's valuation of the debt, is given by

$$PD_1 = \hat{\theta} \times 1 + (1 - \hat{\theta}) \frac{X^L}{F}. \tag{B8}$$

Since the firm has to sell $F = \frac{I}{PD_1}$ units of straight debt to raise an amount of I, multiplying each side of (B8) by $\frac{I}{PD_1}$ yields:

$$I = \hat{\theta} \frac{I}{PD_1} + (1 - \hat{\theta}) X^L. \tag{B9}$$

Solving for PD_1, we obtain:

$$PD_1 = \frac{\hat{\theta} I}{I - (1 - \hat{\theta}) X^L}, \tag{B10}$$

or, equivalently,

$$F = \frac{I - (1 - \hat{\theta}) X^L}{\hat{\theta}}. \tag{B11}$$

The payoff to the equity holders of the firm is 0 in the bad state and $(X^H - F)$ in the good state, so the expected payoff to current shareholders of the firm is

$$EU^{Debt} = \hat{\theta}^f \left(X^H - \frac{I - (1 - \hat{\theta})X^L}{\hat{\theta}} \right). \tag{B12}$$

Now, assume that the firm can issue risk-free straight debt, that is, $I \leq X^L$. The payoff to each unit of straight debt is one in every state, so the market price of each unit of risk-free debt is

$$PD_1 = \hat{\theta} + (1 - \hat{\theta}) = 1. \tag{B13}$$

The firm has to sell I/PD_1 units of risk-free straight debt to raise an amount of I. Thus, $F = I/PD_1 = I$. The payoff to the equity holders of the firm is $(X^L - I)$ in the bad state and $(X^H - I)$ in the good state, so the expected payoff to current shareholders of the firm is

$$EU^{RiskFreeDebt} = \theta^f X^H + (1 - \theta^f)X^L - I. \tag{B14}$$

Q.E.D.

Proof of Lemma 3
We impose the condition that not all investors prefer to convert to equity at time 2 if the bad state with low cash flow (X^L) is realized. Otherwise, there would be no difference between convertible debt and equity. Thus, we have the following restriction on the conversion ratio x and the price p:

$$x\frac{X^L}{1 + x\frac{I}{p}} < 1, \tag{B15}$$

which translates into the following no-conversion condition:[45]

$$F = \frac{I}{p} > X^L - \frac{1}{x}. \tag{B16}$$

In addition, we impose the restriction that all convertible debt investors prefer to convert to equity at time 2 if the good state with high cash flow (X^H) is realized. Otherwise, there would be no difference between convertible debt and straight debt. Thus, we have the following restriction on the conversion ratio x and the price p:

$$F = \frac{I}{p} < X^H - \frac{1}{x}. \tag{B17}$$

Combining these two conversion conditions, we obtain the following restriction:

$$X^L < \frac{I}{p} + \frac{1}{x} < X^H. \tag{B18}$$

First, consider the case in which there is no default of the convertible at time 2. Then the following condition has to be satisfied:

$$F = \frac{I}{p} \leq X^L. \tag{B19}$$

Thus, in the case of risk-free convertible, the valuation of the marginal investor for the convertible security at time 1 is given by

$$p = \hat{\theta} \left(\frac{x}{1 + x\frac{I}{p}} \right) X^H + (1 - \hat{\theta}) \times 1. \tag{B20}$$

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[45] This condition also guarantees that none of the convertible debt investors has the incentive to convert to equity at time 2 if the cash flow is equal to X^L, that is, it is equivalent to this condition: $x\frac{[X^L - (\frac{I}{p} - 1)]}{1 + x} < 1$.

If we solve this equation for the conversion ratio x, we get

$$x = \frac{p^2 - (1 - \hat{\theta})p}{\hat{\theta}X^H p + (1 - \hat{\theta})I - pI}. \tag{B21}$$

From (B21), it follows that x is increasing in p. If we substitute (B21) for x in (B17), (B17) is satisfied as an equality if and only if $p = 1$. Thus, in the case of risk-free convertible debt, the conversion condition (B17) in the good state will be satisfied strictly as an inequality if only if $p > 1$ (since $(X^H - 1/x)$ is increasing in p, and $F = I/p$ is decreasing in p), or, equivalently, if and only if $F = I/p < I$. The expected payoff to the firm's current shareholders (from issuing risk-free convertible debt) based on the insider belief θ^f is

$$EU^{RiskFreeConvertible} = \theta^f \left(\frac{1}{1 + x\frac{I}{p}} \right) X^H + (1 - \theta^f) \left(X^L - \frac{I}{p} \right). \tag{B22}$$

If we plug in the value of x from (B21) in (B22), we obtain

$$EU^{RiskFreeConvertible} = \theta^f X^H + (1 - \theta^f) X^L - \frac{\theta^f}{\hat{\theta}}I + \frac{\left(\theta^f - \hat{\theta} \right)I}{\hat{\theta}p}. \tag{B23}$$

If $\hat{\theta} < \theta^f$, it is optimal to set p as low as possible, that is, $x = \underline{x}$. Given the no-default condition in (B19) and the conversion condition (B17) in the good state of the world, the lowest possible value of p, which we denote by $\underline{p}$, can be determined as follows.

If the investment requirement I is small so that the firm is able to issue risk-free straight debt (i.e., if $I \leq X^L$), the conversion condition (B17) implies that the greatest lower bound on the price p of the risk-free convertible is equal to 1, that is, the lowest possible value of p, $\underline{p}$, must be strictly greater than one as we showed earlier. In other words, the optimal total face value $F = \frac{I}{p}$ of the risk-free convertible must be slightly less than I. The no-default condition (B19) is not binding in this case, since $F < I \leq X^L$.

On the other hand, if the investment requirement I is large so that $I > X^L$, it follows that $\frac{I}{X^L} > 1$. From the no-default condition in (B19) ($F = I/p \leq X^L$), which implies that the total face value F of the risk-free convertible issue must be less than or equal to X^L, it also follows that $\frac{I}{X^L} \leq p$. Thus, the lowest possible value $\underline{p}$ of the convertible price p is $\frac{I}{X^L}$, and it is strictly greater than one. Hence, the no-default condition (B19) implies that in this case, p is optimally set equal to $\underline{p} = \frac{I}{X^L} > 1$. The conversion condition (B17) is not binding in this case, since $F = I/p = X^L < I$.

If we plug in these lower bounds of p in (B21), we obtain the following optimal conversion ratio in each case:

$$\underline{x} = \begin{cases} \dfrac{1}{X^H - I} + \epsilon, & I \leq X^L, \\[2ex] \dfrac{I - (1 - \hat{\theta})X^L}{X^L \left[\hat{\theta}X^H + (1 - \hat{\theta})X^L - I \right]}, & I > X^L, \end{cases} \tag{B24}$$

where ϵ is an arbitrarily small positive number.

Let's first consider the case in which $I > X^L$. Note that since the optimal conversion ratio $x = \underline{x}$ from (B24) implies that $p = \underline{p} = \frac{I}{X^L}$ in this case, it follows from (B23) that

$$EU^{RiskFreeConvertible} = \frac{\theta^f}{\hat{\theta}} (\hat{\theta}X^H + (1 - \hat{\theta})X^L - I) = EU^{Debt}. \tag{B25}$$

Note that in this case, the total face value F of the risk-free convertible debt issue is optimally set to $\frac{I}{p} = X^L$, that is, the firm's cash flow at the bad state.

In the other case in which $I \leq X^L$, we have $p = \underline{p} > 1$. Therefore, it follows from (B23) that

$$EU^{RiskFreeConvertible} < \theta^f X^H + (1 - \theta^f) X^L - I = EU^{RiskFreeDebt}. \tag{B26}$$

In both cases, the price $p = \underline{p}$ of the convertible debt as a function of $\underline{x}$ (by rearranging the pricing equation (B20)) is given by

$$\underline{p} = \frac{(1 - \hat{\theta}) - \underline{x}I + \underline{x}\hat{\theta}X^H + \sqrt{(1 - \hat{\theta} - \underline{x}I + \underline{x}\hat{\theta}X^H)^2 + 4(1 - \hat{\theta})\underline{x}I}}{2}. \tag{B27}$$

Thus, we showed that risk-free straight debt always dominates risk-free convertible debt, if the marginal investor is more pessimistic than firm insiders. If the required investment I is greater than X^L, we also showed that risky straight debt generates the same expected payoff to firm insiders as the risk-free convertible debt ($F = X^L$), when the marginal investor is more pessimistic than firm insiders.

If $\hat{\theta} \geq \theta^f$, it follows from (B23) that it is optimal to set p as high as possible, that is, $x = \overline{x}$. Given the no-conversion condition (in the low cash flow state) from (B16) and the value of x as a function of p from (B21), the highest possible value of p should be less than $\frac{\hat{\theta}X^H + (1 - \hat{\theta})X^L}{X^L}$ (equivalently, $F > \frac{IX^L}{\hat{\theta}X^H + (1 - \hat{\theta})X^L}$). If we plug in this upper bound of p in (B21), we obtain the following optimal conversion ratio:

$$\overline{x} = \frac{\hat{\theta}X^H + (1 - \hat{\theta})X^L}{X^L \left(\hat{\theta}X^H + (1 - \hat{\theta})X^L - I \right)} - \epsilon, \tag{B28}$$

where ϵ is a small positive number. Note that since we have $p < \frac{\hat{\theta}X^H + (1 - \hat{\theta})X^L}{X^L}$, it follows from (B23) that

$$EU^{RiskFreeConvertible} < \theta^f X^H + (1 - \theta^f) X^L - I \frac{\theta^f X^H + (1 - \theta^f) X^L}{\hat{\theta}X^H + (1 - \hat{\theta})X^L} = EU^{Equity}. \tag{B29}$$

Thus, if the marginal investor is more optimistic than firm insiders, issuing equity only is preferred to issuing risk-free convertible debt. The price $p = \overline{p}$ of the convertible debt as a function of $\overline{x}$ (by rearranging the pricing equation (B20)) is given by

$$\overline{p} = (1 - \hat{\theta}) - \overline{x}I + \overline{x}\hat{\theta}X^H + \sqrt{(1 - \hat{\theta} - \frac{\overline{x}I + \overline{x}\hat{\theta}X^H)^2 + 4(1 - \hat{\theta})\overline{x}I}{2}}. \tag{B30}$$

Note that when $I > X^L$, if we allow the convertible debt to default at time 2, when the bad state is realized, we will have $F = \frac{I}{p} > X^L$ (i.e., $p < I/X^L$, see also (B24)). Thus, in the case of risky convertible debt, the valuation of the marginal investor for the convertible security at time 1 will be given by

$$I = \hat{\theta} \left(\frac{xF}{1 + xF} \right) X^H + (1 - \hat{\theta})X^L,$$

$$p = \hat{\theta} \left(\frac{x}{1 + x\frac{I}{p}} \right) X^H + (1 - \hat{\theta}) \frac{X^L}{I/p}, \tag{B31}$$

which leads to

$$\frac{x}{p} = \frac{I - (1 - \hat{\theta})X^L}{I \left[\hat{\theta}X^H + (1 - \hat{\theta})X^L - I \right]}. \tag{B32}$$

From (B32), it follows that $F = \frac{I}{p} = \frac{I-(1-\hat{\theta})X^L}{x[\hat{\theta}X^H+(1-\hat{\theta})X^L-I]}$. Therefore, the possibility of default ($F > X^L$) in the bad state implies that $x < \frac{I-(1-\hat{\theta})X^L}{X^L[\hat{\theta}X^H+(1-\hat{\theta})X^L-I]}$.

From (B32) and the conversion condition (B17) in the good state, it also follows that the conversion ratio x must be higher than $\frac{\hat{\theta}}{\hat{\theta}X^H+(1-\hat{\theta})X^L-I}$. Equivalently, the total face value of the risky convertible debt issue must be strictly less than the total face value of the risky straight debt issue, that is, $F < \frac{I-(1-\hat{\theta})X^L}{\hat{\theta}}$ $\left(p > \frac{\hat{\theta}}{I-(1-\hat{\theta})X^L}\right)$. The expected payoff to the firm's current shareholders (from issuing risky convertible debt) based on the insider belief θ^f is

$$EU^{Convertible} = \theta^f \left(\frac{1}{1+x\frac{I}{p}}\right)X^H = \theta^f \left(\frac{1}{1+I\frac{x}{p}}\right)X^H. \tag{B33}$$

Plugging in the value of $\frac{x}{p}$ from (B32) in (B33), we obtain:

$$EU^{Convertible} = \frac{\theta^f}{\hat{\theta}}\left(\hat{\theta}X^H + (1-\hat{\theta})X^L - I\right) = EU^{Debt}, \tag{B34}$$

for any value of $F \in \left(X^L, \frac{I-(1-\hat{\theta})X^L}{\hat{\theta}}\right)$. Note that the firm's expected payoff from issuing risky convertible debt alone is equal to its expected payoff from issuing risky straight debt alone. Together, (B34) and (B29) also imply that issuing equity alone dominates issuing risky convertible as well, if the marginal investor is more optimistic than firm insiders. Q.E.D.

Proof of Proposition 1

From Equations (B7) and (B12), we obtain

$$EU^{Equity} - EU^{Debt} = \left(1 - \frac{I}{\hat{\theta}X^H + (1-\hat{\theta})X^L}\right)[\theta^f X^H + (1-\theta^f)X^L] - \theta^f\left(X^H - \frac{I-(1-\hat{\theta})X^L}{\hat{\theta}}\right).$$

The firm will prefer issuing equity alone to issuing straight debt alone if and only if $EU^{Equity} - EU^{Debt} > 0$, which is equivalent to

$$\frac{\hat{\theta}X^H + (1-\hat{\theta})X^L - I}{\hat{\theta}X^H + (1-\hat{\theta})X^L}[\theta^f X^H + (1-\theta^f)X^L] > \theta^f \frac{\hat{\theta}X^H + (1-\hat{\theta})X^L - I}{\hat{\theta}}. \tag{B35}$$

If $\hat{\theta}X^H + (1-\hat{\theta})X^L - I > 0$, this condition is equivalent to $\theta^f \geq \hat{\theta}$.

If $\theta^f \geq \hat{\theta}$, we proved that issuing convertible debt alone is dominated by issuing straight debt alone (see Equation (B25)). If $\theta^f < \hat{\theta}$, we proved that issuing convertible debt alone is dominated by issuing equity alone (see Equation (B29)). Therefore, issuing convertible debt alone is never optimal. Q.E.D.

Proof of Proposition 2

First, suppose that the firm issues risk-free debt so that $0 \leq I_D \leq X^L$. The valuation of the entire equity of the firm to an agent with belief θ is then equal to $\theta X^H + (1-\theta)X^L - I_D$. The firm needs to raise $I_E = I - I_D$ by issuing equity. The marginal equity investor $\hat{\theta}_E$ is characterized by

$$\int_{\hat{\theta}_E}^{\theta^m + d} \frac{W}{2d}d\theta = I_E = I - I_D,$$

or, equivalently,

$$\hat{\theta}_E = \theta^m + d\left(1 - \frac{2(I - I_D)}{W}\right) = \hat{\theta} + \frac{2dI_D}{W}, \tag{B36}$$

where $\hat{\theta} = \theta^m + d(1 - \frac{2I}{W})$. Investors in the range $[\hat{\theta}, \hat{\theta}_E]$ will purchase the risk-free debt, which is worth I_D regardless of investor beliefs. The marginal equity investor values the entire equity of the firm at $\hat{\theta}_E X^H + (1 - \hat{\theta}_E)X^L - I_D$. Suppose the firm needs to issue x shares of new equity to raise $I - I_D$, then we have

$$\frac{x}{1 + x} = \frac{I - I_D}{\hat{\theta}_E X^H + (1 - \hat{\theta}_E)X^L - I_D},$$

or, equivalently,

$$x = \frac{I - I_D}{\hat{\theta}_E X^H + (1 - \hat{\theta}_E)X^L - I}. \tag{B37}$$

The expected payoff of firm insiders will be

$$EU^{Combi} = \frac{1}{1 + x}[\theta^f X^H + (1 - \theta^f)X^L - I_D] = \frac{\hat{\theta}_E X^H + (1 - \hat{\theta}_E)X^L - I}{\hat{\theta}_E X^H + (1 - \hat{\theta}_E)X^L - I_D}[\theta^f X^H + (1 - \theta^f)X^L - I_D]. \tag{B38}$$

The objective of firm insiders is to choose the optimal amount of debt, I_D^*, to be issued:

$$\underset{I_D \in [0, X^L]}{\text{Max}} \frac{\hat{\theta}_E X^H + (1 - \hat{\theta}_E)X^L - I}{\hat{\theta}_E X^H + (1 - \hat{\theta}_E)X^L - I_D}[\theta^f X^H + (1 - \theta^f)X^L - I_D]$$

$$s.t. \hat{\theta}_E = \hat{\theta} + \frac{2dI_D}{W}.$$

We have the following F.O.C. to this maximization problem:

$$\frac{\partial EU^{Combi}}{\partial I_D} = \frac{-\hat{X}^2 - 2aI_D\hat{X} + a(1 - a)I_D^2 + X^f\hat{X} + I\hat{X} - (1 - a)IX^f}{[\hat{X} - (1 - a)I_D]^2} = 0,$$

$$= \frac{a[\hat{X} - (1 - a)I_D]^2 - [\hat{X} - (1 - a)X^f][\hat{X} - (1 - a)I]}{(1 - a)[\hat{X} - (1 - a)I_D]^2} = 0.$$

Thus, the optimal amount of debt to be raised is

$$I_D^* = \frac{a\hat{X} - \sqrt{a[\hat{X} - (1 - a)X^f][\hat{X} - (1 - a)I]}}{a - a^2}, \tag{B39}$$

where

$$\hat{X} = \hat{\theta} X^H + (1 - \hat{\theta})X^L, \quad X^f = \theta^f X^H + (1 - \theta^f)X^L, \quad a = \frac{2d(X^H - X^L)}{W} > 0.$$

Note that if $I_D^* \leq 0$, we obtain the corner solution of issuing equity only. In particular, issuing equity alone dominates issuing a combination of equity and risk-free debt if and only of $I_D^* \leq 0$ or, equivalently, if and only if

$$\hat{X}^2 - (I + X^f)\hat{X} + (1 - a)IX^f \geq 0. \tag{B40}$$

If we rearrange this inequality, we obtain the following equivalent restriction in terms of $\hat{\theta}$: $\hat{\theta} \geq \theta_1$, where θ_1 is given by

$$\theta_1 = \frac{-b_1 - \sqrt{-b_1^2 - 4a_1c_1}}{2a_1}, \tag{B41}$$

where

$$a_1 = (X^H - X^L)^2,$$

$$b_1 = -((X^H - X^L)(I + X^f - 2X^L)),$$

$$c_1 = (1 - a)IX^f - X^L(I + X^f - X^L).$$

Note that if the firm issues risk-free debt, it holds that $I_D \leq X_L < I$ by assumption. Therefore, issuing risk-free debt alone to raise I is infeasible. Note also that when we solve for the optimal amount of risk-free debt to be issued in the above problem, we also have to consider the constraint that $I_D^* \leq X^L$. When we plug in the unconstrained optimum I_D^* from (B39) into this constraint, we obtain the following equivalent restriction in terms of $\hat{\theta}$: $\hat{\theta} \geq \theta_2^a$, where θ_2^a is given by

$$\theta_2^a = \frac{-b_2 - \sqrt{-b_2^2 - 4a_2 c_2}}{2a_2}, \tag{B42}$$

where

$$a_2 = (X^H - X^L)^2,$$

$$b_2 = (X^H - X^L)(2(1 + a)X^L - X^f - I),$$

$$c_2 = (X^L)^2 + X^L(2aX^L - X^f - I) - (1 - a)(a(X^L)^2 - IX^f).$$

In other words, if $\hat{\theta} \leq \theta_2^a$, the solution to the above maximization problem is to set $I_D^* = X^L$. Further, the restriction $\hat{\theta} \leq \theta_2^a$ is also equivalent to the restriction $I \geq I_{2a}$, where the investment threshold I_{2a} is defined by:

$$I_{2a} = \frac{4d^2(X^H - X^L)(W + 2X^L) + dW(W - 4(X^L + \theta^m(X^H - X^L))) - W(W(\theta^f - \theta^m) + \sqrt{U})}{4d(W + 2d(X^H - X^L))}, \tag{B43}$$

where $U = 16\theta^f(\theta^m + d)d^2(X^H - X^L)^2 + W^2(\theta^m + d - \theta^f)^2$. Note also that $\theta_2^a = \theta^m + d(1 - \frac{2I_{2a}}{W})$.

Now we consider the case in which it is optimal for the firm to issue risky debt. Suppose that the firm raises an amount $I_E = I - I_D$ by issuing equity, and an amount I_D by issuing risky debt so that $I_D > X^L$. The face value of the debt F is then given by

$$F = \frac{I_D - (1 - \hat{\theta})X^L}{\hat{\theta}}. \tag{B44}$$

The marginal equity investor is characterized by his belief $\hat{\theta}_E = \hat{\theta} + \frac{2dI_D}{W}$. Investors in the range $[\hat{\theta}, \hat{\theta}_E]$ will purchase the risky debt, and the price of this debt will be determined by the marginal debt investor with belief $\hat{\theta}$. The total market value of equity in this case is $\hat{\theta}_E[X^H - F]$. Suppose the firm needs to issue x shares of new equity to raise $I - I_D$, then we have

$$\frac{x}{1 + x} = \frac{I - I_D}{\hat{\theta}_E[X^H - F]}, \tag{B45}$$

or, equivalently,

$$x = \frac{I - I_D}{\hat{\theta}_E[X^H - F] - (I - I_D)}. \tag{B46}$$

Therefore, the expected payoff of firm insiders is equal to

$$EU^{Combi} = \left(1 - \frac{I - I_D}{\hat{\theta}_E(X^H - F)}\right)\theta^f(X^H - F). \tag{B47}$$

The firm has to choose the optimal split between debt and equity, that is,

$$\max_{I_D \in [X^L, I]} \left(1 - \frac{I - I_D}{\hat{\theta}_E(X^H - F)}\right)\theta^f(X^H - F)$$

$$\text{s.t. } \hat{\theta}_E = \hat{\theta} + \frac{2dI_D}{W} \text{ and } F = \frac{I_D - (1 - \hat{\theta})X^L}{\hat{\theta}}. \tag{B48}$$

The solution to this maximization problem is

$$I_D^* = \frac{\sqrt{\hat{\theta}W(\hat{\theta}W + 2dI)} - \hat{\theta}W}{2d}. \tag{B49}$$

Note that it is never true that $I_D^* \geq I$ where I_D^* is given by (B49). Therefore, issuing risky debt alone is always dominated by issuing a combination of equity and risky debt. Note also that an interior optimal solution to this problem is obtained only if the constraint $I_D^* \geq X^L$ does not bind. Given (B49), this leads to the following necessary condition for the optimality of an interior solution:

$$2dI^2 - [(\theta^m + d)W + 4dX^L]I + 2X^L((\theta^m + d)W + dX^L) > 0. \tag{B50}$$

Equivalently,

$$I > I_{2b} = \frac{(\theta^m + d)W + 4dX^L - \sqrt{[(\theta^m + d)W + 4dX^L]^2 - 16dX^L((\theta^m + d)W + dX^L)}}{4d}. \tag{B51}$$

Thus, if $I \leq I_{2b}$, it is optimal to set $I_D^* = X^L$ in the above maximization problem given in (B48). Equivalently, the optimization problem given in (B48) will have an interior solution ($I_D^* > X^L$), if and only if $\hat{\theta} < \theta_2^b$, where

$$\theta_2^b = \theta^m + d\left(1 - \frac{2I_{2b}}{W}\right). \tag{B52}$$

Whether it is optimal to issue risky or risk-free debt in combination with equity depends on the comparison of the belief thresholds θ_2^a and θ_2^b. If $\theta_2^b \leq \theta_2^a$, it is optimal to issue risk-free debt in combination with equity if $\theta_1 > \hat{\theta} \geq \theta_2^a$ and risky debt in combination with equity if $\hat{\theta} < \theta_2^b$. In particular, in this case, if $\theta_2^b \leq \hat{\theta} \leq \theta_2^a$, it is optimal to set $I_D = X^L$ and issue risk-free debt in combination with equity. On the other hand, if $\theta_2^b > \theta_2^a$, it is optimal to issue risk-free debt if $\theta_1 > \hat{\theta} \geq \theta_2^b$ and risky debt if $\hat{\theta} < \theta_2^a$. However, if $\theta_2^a \leq \hat{\theta} < \theta_2^b$, the optimality of issuing risky debt or risk-free debt in combination with equity depends on the comparison of the value functions of the two maximization problems above. In fact, in this case, the continuity of the value functions implies that there exists an indifference threshold belief $\theta_2^c \in [\theta_2^a, \theta_2^b)$, such that it is optimal to issue risk-free debt in combination with equity if $\theta_1 > \hat{\theta} \geq \theta_2^c$, and risky debt in combination with equity if $\hat{\theta} < \theta_2^c$. The definition of θ_2 is then as follows. If $\theta_2^b \leq \theta_2^a$, we have $\theta_2 = \theta_2^b$ and otherwise $\theta_2 = \theta_2^c$.

$$\theta_2 = \begin{cases} \theta_2^b & \text{if } \theta_2^b \leq \theta_2^a, \\ \theta_2^c & \text{if } \theta_2^b > \theta_2^a. \end{cases} \tag{B53}$$

When the firm issues convertible debt, let us assume that the firm chooses the conversion ratio optimally so that the face value of the convertible debt is F^*. The price of the convertible debt (both the equity component and the straight debt component) is determined by the valuation of the marginal investor $\hat{\theta}$. Now assume that the firm issues F^* units of straight debt and issues equity to raise the remaining amount, $I - PD_1 \times F^*$. The price of straight debt is determined by the valuation of the marginal investor $\hat{\theta}$, while the price of equity is determined by the marginal investor $\hat{\theta}_E = \hat{\theta} + \frac{2d \times PD_1 \times F^*}{W} > \hat{\theta}$.

If $\hat\theta \le \theta^f$, we showed in lemma 3 that F^* can take any value in the interval $[X^L, \frac{I-(1-\hat\theta)X^L}{\hat\theta})$. Without loss of generality, set $F^* = X^L$. Thus, $PD_1 = 1$. Then, we know from lemma 3 that the expected payoff to the current shareholders from issuing convertible debt only is equal to

$$EU^{convertible} = \frac{\theta^f}{\hat\theta}(\hat\theta X^H + (1-\hat\theta)X^L - I), \tag{B54}$$

$$= \theta^f(X^H - X^L) - \frac{\theta^f}{\hat\theta}(I - X^L). \tag{B55}$$

If the firm issues a combination of risk free debt and equity, with the amount of debt issued equal to $I_D = X^L$, the expected payoff to the current shareholders (from proposition 2) is given by

$$EU^{combined} = \left(1 - \frac{I - X^L}{\hat\theta X^H + (1-\hat\theta)X^L - (1-a)X^L}\right)\theta^f(X^H - X^L), \tag{B56}$$

$$= \theta^f(X^H - X^L) - \frac{\theta^f}{(\hat\theta + 2d\frac{X^L}{W})}(I - X^L). \tag{B57}$$

Thus, if $\hat\theta \le \theta^f$, $EU^{combined} > EU^{convertible}$. Thus, the firm can sell the equity component at a higher price to outside investors if it issues a debt-equity combination rather than convertible debt only. If $\hat\theta > \theta^f$, we know from lemma 3 that issuing equity alone dominates issuing convertible debt alone. Moreover, in this case, it follows from proposition 2 that, if $\hat\theta < \theta_1$, issuing a combination of debt and equity dominates issuing equity alone. Thus, $EU^{combined} > EU^{convertible}$, if $\hat\theta > \theta^f$ as well.
Q.E.D.

Proof of Proposition 3
First, consider the case in which the firm issues a combination of two tranches of equity and risk-free debt, so that $0 \le I_D \le X^L$ (equivalently, $I - X^L \le I_E \le I$). For a given level of total equity I_E issued, the firm's objective given in (9) is clearly equivalent to

$$\max_{I_1} \left(1 - \frac{I_1}{X_{\hat\theta_1} - (I - I_E)} - \frac{I_E - I_1}{X_{\hat\theta_E} - (I - I_E)}\right). \tag{B58}$$

Differentiating this objective function with respect to I_1 and setting the result to zero, we obtain the following first order condition:

$$-\frac{2dI_1(X^H - X^L)}{W(X^L + \theta_1(X^H - X^L) - (I - I_E))^2} - \frac{1}{X^L + \theta_1(X^H - X^L) - (I - I_E)} + \frac{1}{X^L + \theta_E(X^H - X^L) - (I - I_E)} = 0. \tag{B59}$$

Solving this equation for I_1, we obtain

$$I_1^* = \frac{W(X_{md} - I_D) - \sqrt{W(X_{md} - I_D)(W(X_{md} - I_D) - 2dI_E(X^H - X^L))}}{2d(X^H - X^L)}, \tag{B60}$$

where $X_{md} = (\theta^m + d)X^H + (1 - (\theta^m + d))X^L$. We also note that the second derivative of the objective function w.r.t. I_1 is strictly negative. It follows from (B60) that if $I_D = 0$, so that the firm issues two tranches of equity alone to finance the project, then the optimal breakpoint of equity tranching is given by:

$$I_1^* = \frac{WX_{md} - \sqrt{WX_{md}(WX_{md} - 2dI(X^H - X^L))}}{2d(X^H - X^L)}. \tag{B61}$$

Note that from proposition 2, we know that if $\hat{\theta} \geq \theta_1$, where θ_1 is given in (B41), then it is optimal for the firm to issue untranched equity only rather than to issue a combination of untranched equity and untranched risk-free debt. Since (B61) implies that $0 < I_1^* < I$, it follows that issuing two tranches of equity alone dominates issuing untranched equity alone. Thus, given that the ability to issue tranches of equity increases the firm insiders' expected payoff from issuing equity for any level of the marginal outside investors' belief, it must also be the case that if $\hat{\theta} \geq \theta_1$, the firm will prefer to issue two tranches of equity alone rather than to issue a combination of two tranches of equity and risk-free debt so that $I_E^* = I$ in this case.

To fully solve the firm's problem when the firm issues a combination of two tranches of equity and risk-free debt, we substitute I_1^* from (B60) into the objective function given in (9) and obtain the following equivalent objective function:

$$\max_{I_E} \left(1 - \frac{2WI_E}{T - 2dI_E(X^H - X^L) + \sqrt{T(T - 2dI_E(X^H - X^L))}}\right)(\theta^f X^H + (1 - \theta^f)X^L - (I - I_E)),$$

(B62)

where $T = W(X^L + (\theta^m + d)(X^H - X^L) - (I - I_E)))$. Setting the derivative of this objective function with respect to I_E to zero, we obtain its first-order condition, which is given by:

$$1 - \frac{2W(\theta^f X^H + (1 - \theta^f)X^L - I + 2I_E)}{T - 2dI_E(X^H - X^L) + \sqrt{T(T - 2dI_E(X^H - X^L))}}$$
$$+ \frac{2WI_E(\theta^f X^H + (1 - \theta^f)X^L - I + I_E)(W - 2d(X^H - X^L) + N)}{\left(T - 2dI_E(X^H - X^L) + \sqrt{T(T - 2dI_E(X^H - X^L))}\right)^2} = 0,$$

(B63)

where

$$N = \frac{W((W - 2d(X^H - X^L))(X^L + (\theta^m + d)(X^H - X^L) - (I - I_E)) + T - 2dI_E(X^H - X^L))}{2\sqrt{T(T - 2dI_E(X^H - X^L))}}.$$

(B64)

This FOC finally simplifies into a quartic equation (an equation of the fourth degree) in I_E, and the optimal solution I_E^* is equal to one of the real roots of this quartic equation.

Second, consider the case in which the firm issues a combination of two tranches of equity and *risky* debt, so that $I_D > X^L$ (equivalently, $0 \leq I_E < I - X^L$). For a given level of total equity I_E issued, the firm's objective given in (10) is clearly equivalent to

$$\max_{I_1} \left(1 - \frac{I_1}{\hat{\theta}_1(X^H - F)} - \frac{I_E - I_1}{\hat{\theta}_E(X^H - F)}\right),$$

(B65)

where $F = \frac{I - I_E - (1 - \hat{\theta})X^L}{\hat{\theta}}$. Differentiating this objective function with respect to I_1 and setting the result to zero, we obtain the following first-order condition:

$$\frac{2dW(W(\theta^m + d) - 2dI)(W(\theta^m + d)(I_E - 2I_1) + 2dI_1^2)}{(W(\theta^m + d) - 2dI_1)^2(W(\theta^m + d) - 2dI_E)(W(X^L + (\theta^m + d)(X^H - X^L) - (I - I_E)) - 2dI(X^H - X^L))} = 0.$$

(B66)

Solving this equation for I_1, we obtain

$$I_1^* = \frac{W(\theta^m + d) - \sqrt{W(\theta^m + d)(W(\theta^m + d) - 2I_E)}}{2d}.$$

(B67)

We also note that, in this case, the second derivative of the objective function w.r.t. I_1 is strictly negative. Next, we substitute I_1^* from (B67) into the objective function in (10) and obtain the following equivalent objective function:

$$\max_{I_E} \theta^f \left(X^H - X^L + \frac{W}{d} - \frac{I - X^L - I_E}{\theta^m + d} - \frac{2dI(I - X^L - I_E)}{(\theta^m + d)(W(\theta^m + d) - 2dI)} \right)$$
$$- \theta^f \left(\frac{2WI_E}{\sqrt{W(\theta^m + d)(W(\theta^m + d) - 2dI_E)}} + \frac{\sqrt{W(\theta^m + d)(W(\theta^m + d) - 2dI_E)}}{d(\theta^m + d)} \right), \quad \text{(B68)}$$

Setting the derivative of this objective function with respect to I_E to zero, we obtain the following FOC:

$$\theta^f W \left(\frac{1}{W(\theta^m + d) - 2dI} - \frac{\sqrt{W(\theta^m + d)(W(\theta^m + d) - 2dI_E)}}{(W(\theta^m + d) - 2dI_E)^2} \right) = 0. \quad \text{(B69)}$$

Solving this FOC for I_E, the optimal amount of total equity issued is given by:

$$I_E^* = \frac{W(\theta^m + d) - \sqrt[3]{W(\theta^m + d)(W(\theta^m + d) - 2dI)^2}}{2d}. \quad \text{(B70)}$$

Thus, issuing a combination of two tranches of equity and untranched risky debt is optimal if and only if $I_E^* < I - X^L$, where I_E^* is given by (B70). This condition is equivalent to $I > I_{eq}^t$, where the investment threshold I_{eq}^t is defined by the following indifference equation in I:

$$((\theta^m + d)W - 2d(I - X^L))^3 - (\theta^m + d)W((\theta^m + d)W - 2dI)^2 = 0. \quad \text{(B71)}$$

The threshold I_{eq}^t is equal to the smallest real root of this cubic equation in I, and the threshold belief θ_e^t is defined by

$$\theta_e^t = \theta^m + d \left(1 - \frac{2I_{eq}^t}{W} \right). \quad \text{(B72)}$$

Thus, if $\hat{\theta} < \theta_e^t$ ($I > I_{eq}^t$), the firm will find it optimal to issue two tranches of equity in combination with untranched risky debt rather than to issue two tranches of equity in combination with untranched risk-free debt. On the other hand, if $\hat{\theta} \geq \theta_e^t$, it is optimal for the firm to issue a combination of two tranches of equity and untranched risk-free debt.

From (B60) and (B67), it clearly follows that for any $I_E = I - I_D$, it holds that $0 < I_1^* < I_E$. Therefore, issuing a combination of two tranches of equity and untranched debt yields a higher expected payoff to firm insiders than issuing a combination of untranched equity and untranched debt. Q.E.D.

Proof of Proposition 4

First, consider the case in which the firm issues a combination of equity and risk-free debt, so that $0 \leq I_D \leq X^L$ or, equivalently, $I - X^L \leq I_E \leq I$. Then, the face value of the debt is equal to $I_D = I - I_E$. The total value of equity is then

$$\hat{\theta}_E X^H + (1 - \hat{\theta}_E)X^L - (I - I_E).$$

Firm insiders will choose the optimal level of I_E to maximize their expected payoff from issuing a combination of equity and risk-free debt:

$$\max_{I_E \in [I - X^L, I]} \left(1 - \frac{I_E}{\hat{\theta}_E X^H + (1 - \hat{\theta}_E)X^L - (I - I_E)} \right) (\theta^f X^H + (1 - \theta^f)X^L - (I - I_E)), \quad \text{(B73)}$$

where $\hat{\theta}_E = \theta^m + d\left(1 - \frac{2I_E}{W}\right)$. We first assume that the constraint $I_E \in [I - X^L, I]$ is satisfied. After differentiating the objective function with respect to I_E and setting it to zero, we obtain the following unconstrained optimum:

$$I_E^* = \frac{-2dW((\theta^m + d)(X^H - X^L) + X^L - I) + \sqrt{2AB}}{2d(W - 2d(X^H - X^L))}, \tag{B74}$$

where

$$A = dW^2(X^L + (\theta^m + d)(X^H - X^L) - I),$$

$$B = 2d(X^L + \theta^f(X^H - X^L) - I) + W(\theta^m + d - \theta^f).$$

It is straightforward to verify that the second derivative of the objective function in (B73) w.r.t. I_E is negative. Substituting I_E^* from (B74) into (B73), we obtain the value function

$$J_{free}^{eq} = \frac{2d^2(X^H - X^L)^2(W + 2(X^L + \theta^f(X^H - X^L) - I)) + Q - 2\sqrt{2}(X^H - X^L)\sqrt{P}}{(W - 2d(X^H - X^L))^2}, \tag{B75}$$

where

$$P = dW^2(X^L + (\theta^m + d)(X^H - X^L) - I)(d(W + 2(X^L + \theta^f(X^H - X^L) - I)) + W(\theta^m - \theta^f)),$$

$$Q = dW(X^H - X^L)(W - 2(\theta^f - \theta^m)(X^H - X^L)) + W^2(X^L + \theta^m(X^H - X^L) - I).$$

Next, we analyze the conditions under which the firm prefers issuing equity alone to issuing a combination of equity and risk-free debt. This will be the case if $I_E^* \geq I$. This translates into the following equivalent condition for I:

$$I \leq I_1^t = \frac{W(W(\theta^m + d - \theta^f) + 4d(X^L + (\theta^m + d)(X^H - X^L)) - \sqrt{C})}{4d(W + 2d(X^H - X^L))}, \tag{B76}$$

where

$$C = 16d^2(X^L + \theta^f(X^H - X^L))(X^L + (\theta^m + d)(X^H - X^L)) + W^2(\theta^m + d - \theta^f)^2.$$

Thus, if $\hat{\theta} \geq \theta_1^t$, where

$$\theta_1^t = \theta^m + d\left(1 - \frac{2I_1^t}{W}\right), \tag{B77}$$

the firm will find it optimal to issue equity alone rather than to issue a combination of equity and risk-free debt.

Another threshold of interest relates to the case in which $I_E^* \leq I - X^L$. In this case, the firm maxes out its capacity to issue risk-free debt in combination with equity. This translates into the following equivalent condition for I:

$$I \geq I_1^m = \frac{4d^2(X^H - X^L)(W + 2X^L) + dW(W - 4(X^L + \theta^m(X^H - X^L))) - W(W(\theta^f - \theta^m) + \sqrt{U})}{4d(W + 2d(X^H - X^L))}, \tag{B78}$$

where $U = 16\theta^f(\theta^m + d)d^2(X^H - X^L)^2 + W^2(\theta^m + d - \theta^f)^2$. Thus, if $I \geq I_1^m$ or, equivalently, if $\hat{\theta} \leq \theta_1^m$, where

$$\theta_1^m = \theta^m + d\left(1 - \frac{2I_1^m}{W}\right), \tag{B79}$$

it is optimal to set $I_E = I - X^L$ in the above constrained optimization problem given in (B73). However, since $I_E^* < I - X^L$ if $\hat{\theta} < \theta_1^m$, this implies that, in this case, the firm would

have issued less equity than $(I - X^L)$, if it had not faced the constraint $I_E \geq (I - X^L)$. Substituting $I_E = (I - X^L)$ into the objective function (B73), we thus obtain the following value function when $I \geq I_1^m$ ($\hat{\theta} \leq \theta_1^m$) for the above optimization problem in (B73):

$$J_{free}^{eq-corner} = \theta^f \left(X^H - X^L - \frac{W(I - X^L)}{W(\theta^m + d) - 2d(I - X^L)} \right). \tag{B80}$$

Now, we consider the case in which the firm issues a combination of equity and two tranches of debt. This is possible if and only if $I_D = I - I_E > X^L$ or, equivalently, if and only if $I_E < I - X^L$. As we mentioned in the main body of the paper, firm insiders will solve the maximization problem given in (15). If $I_D - X^L \leq I_1 \leq I_D$, the second tranche (worth $I_2 = I_D - I_1$) will be completely risk-free and the total face value of debt will be given by

$$F = F_1 + F_2 = \frac{I_1 - (1 - \hat{\theta}_1)(X^L - (I_D - I_1))}{\hat{\theta}_1} + (I_D - I_1), \tag{B81}$$

where $\hat{\theta}_1 = \theta^m + d\left(1 - \frac{2(I - I_D + I_1)}{W}\right)$. Since, in this case, $\frac{\partial F}{\partial I_1} = \frac{2dW(I_D - X^L)}{(d(W - 2(I - I_D + I_1)) + \theta^m W)^2} > 0$, it follows that it is never optimal to set $I_1 > I_D - X^L$ for any level of I_E where it is feasible to issue two tranches of debt in combination with equity. Thus, the firm optimally sets $I_1^* = I_D - X^L = (I - I_E) - X^L$ and $I_2^* = X^L$ in this case. Then, the face values of the debt tranches are determined according to (13) so that

$$F_1 = \frac{(I - I_E) - X^L}{\hat{\theta}_1}, F_2 = X^L,$$

where $\hat{\theta}_1 = \theta^m + d\left(1 - \frac{2(I - X^L)}{W}\right)$. The firm insiders then solve the following problem:

$$\max_{I_E \in [0, I - X^L)} \left(1 - \frac{I_E}{\hat{\theta}_E(X^H - F)}\right)\theta^f(X^H - F), \tag{B82}$$

where $F = \frac{(I - I_E) - X^L}{\hat{\theta}_1} + X^L$. After differentiating the objective function with respect to I_E and setting it to zero, we obtain the following FOC:

$$\theta^f W \left(\frac{1}{W(\theta^m + d) - 2d(I - X^L)} - \frac{2dI_E}{(W(\theta^m + d) - 2dI_E)^2} - \frac{1}{W(\theta^m + d) - 2dI_E}\right) = 0, \tag{B83}$$

from which we obtain the optimum solution as

$$I_E^* = \frac{2W(\theta^m + d) - \sqrt{(-2W(\theta^m + d))^2 - 8dW(\theta^m + d)(I - X^L)}}{4d}. \tag{B84}$$

The second derivative of the objective function w.r.t. I_E is equal to

$$-\frac{4d\theta^f W^2(\theta^m + d)}{(W(\theta^m + d) - 2dI_E)^3} < 0.$$

Substituting I_E^* from (B84) into the objective function in (B82), we obtain the following value function:

$$J_{debt}^{tr} = \theta^f \left(X^H - X^L + \frac{W}{d} - \frac{W^2(\theta^m + d)}{d\sqrt{W(\theta^m + d)(W(\theta^m + d) - 2d(I - X^L))}} \right). \tag{B85}$$

Subtracting $J_{free}^{eq-corner}$ in (B80) from J_{debt}^{tr}, we obtain:

$$J_{debt}^{tr} - J_{free}^{eq-corner} = \frac{\theta^f W\left(W(\theta^m + d) - d(I - X^L) - \sqrt{W(\theta^m + d)(W(\theta^m + d) - 2d(I - X^L))}\right)}{d(W(\theta^m + d) - 2d(I - X^L))} > 0.$$

(B86)

Therefore, it follows that if $\hat{\theta} \leq \theta_1^m$ ($I \geq I_1^m$), the firm finds it optimal to issue a combination of equity and two tranches of debt (where one of the debt tranches is completely risk free) rather than to issue a combination of equity and risk-free debt. Further, another threshold belief θ_2^t, at which the firm is indifferent between these two choices of security issuance, is defined by solving for that level of $I = I_2^t$, which satisfies the indifference equation

$$J_{debt}^{tr}(I_2^t) = J_{free}^{eq}(I_2^t).$$

(B87)

From this indifference equation, it follows that

$$\theta_2^t = \theta^m + d\left(1 - \frac{2I_2^t}{W}\right).$$

(B88)

Further, given that $J_{debt}^{tr}(I_1^m) - J_{free}^{eq-corner}(I_1^m) > 0$ and from the above derivations of I_1^m and θ_1^m in (B78) and (B79), respectively, it also follows that $I_2^t < I_1^m$ and $\theta_2^t > \theta_1^m$.

Next, we analyze the case in which it is optimal to issue equity in combination with two risky tranches of debt so that the firm prefers to set I_1, the value of the riskier debt tranche, strictly less than $I_D - X^L$. In this case, $I_2 = I_D - I_1 > X^L$ so that the safer debt tranche is also risky. The firm insiders' problem is given by

$$\max_{I_1, I_E \in [0, I - X^L)} \left(1 - \frac{I_E}{\hat{\theta}_E(X^H - F)}\right)\theta^f(X^H - F),$$

(B89)

where $0 \leq I_1 < I_D - X^L$, and the face values of debt tranches are given by (14) so that

$$F = F_1 + F_2 = \frac{I_1}{\hat{\theta}_1} + \frac{(I - I_E) - I_1 - (1 - \hat{\theta})X^L}{\hat{\theta}},$$

(B90)

where $\hat{\theta}_1 = \theta^m + d\left(1 - \frac{2(I_E + I_1)}{W}\right)$. From the objective function in (B89), it follows that for any $I_E \in [0, I - X^L)$, the firm must choose the optimal level of I_1 to minimize the total face value F given in (B90), subject to the constraint $0 \leq I_1 < I_D - X^L$ or, equivalently, $I_2 = I_D - I_1 > X^L$. Differentiating F in (B90) w.r.t. I_1, and setting $\frac{\partial F}{\partial I_1} = 0$, we obtain

$$I_1^* = \frac{-(4d(I - I_D) - 2W(\theta^m + d)) - \sqrt{(4d(I - I_D) - 2W(\theta^m + d))^2 - 8dI_D(W(\theta^m + d) - 2d(I - I_D))}}{4d},$$

(B91)

$$I_2^* = I_D - I_1^* = \frac{-((\theta^m + d)W - 2dI) + \sqrt{((\theta^m + d)W - 2dI)((\theta^m + d)W - 2d(I - I_D))}}{2d}.$$

(B92)

Substituting I_1^* from (B91) in (B90), we obtain

$$F = \frac{d^2 X^L(W - 2I) - dW(W - 2I + (1 - \theta^m)X^L) + W\left(-\theta^m W + \sqrt{((\theta^m + d)W - 2dI)((\theta^m + d)W - 2dI_E)}\right)}{d((\theta^m + d)W - 2dI)}.$$

(B93)

Substituting F from (B93) into the objective function in (B89), and setting its derivative with respect to I_E to zero, we obtain

$$I_E^* = \frac{1}{2}\left(\frac{W(\theta^m + d) - \sqrt[3]{W^2(\theta^m + d)^2(W(\theta^m + d) - 2dI)}}{d}\right), \tag{B94}$$

$$I_D^* = I - I_E^* = \frac{1}{2}\left(\frac{\sqrt[3]{W^2(\theta^m + d)^2(W(\theta^m + d) - 2dI)} - (W(\theta^m + d) - 2dI)}{d}\right). \tag{B95}$$

Substituting I_D^* from (B95) in (B92), the size of the safer risky debt tranche is given by

$$I_2^* = \frac{\sqrt[3]{W(\theta^m + d)(W(\theta^m + d) - 2dI)^2} - (W(\theta^m + d) - 2dI)}{2d}. \tag{B96}$$

Issuing a combination of equity and two risky debt tranches is optimal if and only if $I_2^* > X^L$. This condition is equivalent to $I > I_3^t$, where the investment threshold I_3^t is defined by the following indifference equation in I:

$$((\theta^m + d)W - 2d(I - X^L))^3 - (\theta^m + d)W((\theta^m + d)W - 2dI)^2 = 0. \tag{B97}$$

The threshold I_3^t is equal to the smallest real root of this cubic equation in I, and the threshold belief θ_3^t is defined by

$$\theta_3^t = \theta^m + d\left(1 - \frac{2I_3^t}{W}\right). \tag{B98}$$

Thus, if $\hat{\theta} < \theta_3^t$ ($I > I_3^t$), the firm will find it optimal to issue equity in combination with two risky debt tranches rather than to issue equity in combination with two debt tranches, where one tranche is completely risk-free. On the other hand, if $\theta_3^t \leq \hat{\theta} < \theta_2^t$, it is optimal for the firm to issue a combination of equity and two tranches of debt, where one of the tranches is completely risk free. Note that in all the above cases, $I_E^* > 0$, so that issuing debt alone is never optimal. Further, one should also note that in the above cases in which it is feasible to issue two tranches of debt, that is, when $I_D > X^L$, the optimal solutions implied that $I_1^* > 0$ and $I_2^* = I_D - I_1^* \geq X^L$, so that issuing two tranches of debt dominates issuing untranched risky debt. Q.E.D.

Proof of Proposition 5
When outsiders' average belief is optimistic, and dispersion is high, so that $\theta^f \leq \hat{\theta}$, we have shown in the proof of lemma 3 that $E^{Convertible} < X^f - IX^f\hat{X} = E^{Equity}$. Therefore, equity will dominate convertible debt. We have already shown that equity dominates debt in proposition 1. Hence, it is optimal for the firm to issue equity in this case.

When $\theta^f > \hat{\theta}$, the choice is between convertible debt and straight debt, and which one of these is optimal depends on the required investment financing I. If the firm can issue risk-free straight debt, the costs of financial distress will be zero for both securities. Lemma 3 shows that risk-free convertible debt will be more undervalued than risk-free straight debt. Therefore, the firm will choose to issue risk-free straight debt alone to minimize the undervaluation cost. If the firm's straight debt issue has to be risky since $I > X^L$, we know from lemma 3 that the undervaluation of the risky straight debt is the same as the undervaluation of the risk-free convertible debt with total face value $F = X^L$. Since the firm can minimize the cost of financial distress to zero by setting $F = X^L$, it will

optimally choose to issue risk-free convertible debt with face value $F = X^L$ rather than risky straight debt. Q.E.D.

Proof of Proposition 6

The proof of part 1 is similar to that of proposition 2. One difference relates to the issue cost C^I. Note that the firm now has to incur the issue cost C^I for each tranche of security when it issues a combination of debt and equity. Since C^I is nonzero, the belief threshold θ_{1b} above which the firm finds it optimal to issue equity alone rather than a combination of risk-free debt and equity will be higher compared to the value of θ_1 in proposition 2.

The proof of part 2 is as follows. If the outsiders' average belief about future cash flows is not too high, and the dispersion in outsider beliefs is not too large so that $\hat\theta < \theta_{1b}$ (i.e., $\hat{X}^2 - (I + X^f)\hat{X} + (1 - a)IX^f < 0$, and issue cost C^I is not too high), it is not optimal for the firm to issue equity alone because equity will be too much undervalued by outsiders. In this case, the choice is between issuing an equity-debt combination and issuing debt alone.

First, regardless of the issue cost, the firm has to determine whether it is preferable to issue a combination of risk-free debt and equity or a combination of risky debt and equity. From proposition 2, we know that there exists a threshold belief level θ_2 below which the firm finds it optimal to issue a combination of risky debt and equity, since selling a combination of a large amount of equity and risk-free debt involves high undervaluation costs as the marginal investor becomes less optimistic. But now in the full-fledged model, selling risky debt will also be costly, since the firm expects a positive cost of financial distress C^B in the case of a default when the low state cash flow $X^L < I$ is realized. Therefore, the belief threshold below which the firm finds it optimal to issue a combination of risky debt and equity decreases from θ_2 to θ_{2b}.

Second, once the firm insiders determine which combination of debt and equity is most preferable, they compare it to issuing debt only. We have proved in proposition 2 that issuing a combination of debt and equity dominates issuing debt alone without issue costs. However, with positive issue cost C^I, the firm faces a trade-off. It is optimal for the firm to issue a combination of equity and debt, if the issue cost is low ($C^I \leq \overline{C}_1^I$ if $\theta_{2b} \leq \hat\theta < \theta_{1b}$, and $C^I \leq \overline{C}_2^I$ if $\hat\theta < \theta_{2b}$). Otherwise, the firm will choose to issue straight debt alone when the issue cost is too high, since the additional issue cost associated with the equity-debt combination will outweigh its benefits. Q.E.D.

Proof of Proposition 7

The proof is very similar to the proof of part 2 of proposition 6. The firm faces a choice between issuing a combination of debt and equity and issuing convertible debt if and only if the outsiders' average belief about future cash flows is not too high, and the dispersion in outsider beliefs is not too large so that $\hat\theta < \theta_{1b}$ (i.e., $\hat{X}^2 - (I + X^f)\hat{X} + (1 - a)IX^f < 0$, and issue cost C^I is not too high). In the absence of issue costs and costs of financial distress, we showed in proposition 2 that a combination of equity and straight debt will lead to a higher market valuation of the firm's security issue than convertible debt alone. This is because the firm can sell equity and debt to different groups of investors while the firm is forced to sell the convertible debt to the same group of investors. However, issuing equity-debt combination instead of convertible debt leads to additional issue costs. The firm will choose the optimal security based on this trade-off. As in the proof of proposition 6, we first determine whether it is preferable to issue a combination of risk-free debt and equity or a combination of risky debt and equity (when the issue costs are zero) based on the trade-off between selling undervalued equity and costs of financial distress C^B. Then, the firm compares the preferred debt-equity combination to issuing convertible debt alone. The firm will choose a combination of debt and equity, if the issue cost C^I is low ($C^I \leq \overline{C}_3^I$ if $\theta_{2b} \leq \hat\theta < \theta_{1b}$, and $C^I \leq \overline{C}_4^I$ if $\hat\theta < \theta_{2b}$). Otherwise, the firm will choose to issue convertible debt. Q.E.D.

Proof of Proposition 8

The marginal investor's belief on the firm value is $\theta^m + d$ at time 0. Investors anticipate that the firm will issue E_1 shares of new equity at time 1 to raise the amount I. The price of stock before issuance of equity is therefore

$$PE_0^{Equity} = \frac{1}{1 + E_1}[(\theta^m + d)X^H + (1 - \theta^m - d)X^L], \tag{B99}$$

$$= \left(1 - \frac{I}{\hat{\theta}X^H + (1 - \hat{\theta})X^L}\right)[(\theta^m + d)X^H + (1 - \theta^m - d)X^L]. \tag{B100}$$

After the issuance of equity, the total number of shares outstanding is $1 + E_1$, and the marginal investor is now $\hat{\theta}$ and the equity price per share is

$$PE_1^{Equity} = \frac{1}{1 + E_1}[\hat{\theta}X^H + (1 - \hat{\theta})X^L] = \hat{\theta}X^H + (1 - \hat{\theta})X^L - I. \tag{B101}$$

Therefore, the stock price will go down immediately after more shares come to the market. The price impact is

$$\Delta PE^{Equity} = PE_1^{Equity} - PE_0^{Equity} = -\left(1 - \frac{I}{\hat{\theta}X^H + (1 - \hat{\theta})X^L}\right)\frac{2dI}{W}(X^H - X^L) < 0. \tag{B102}$$

If the firm issues debt, the face value of the debt is $F = \frac{I - (1-\hat{\theta})X^L}{\hat{\theta}}$ and the equity price will be

$$PE_0^{Debt} = PE_1^{Debt} = (\theta^m + d)\left(X^H - \frac{I - (1 - \hat{\theta})X^L}{\hat{\theta}}\right) \tag{B103}$$

at both times 0 and 1. Therefore, the price impact of debt issuance is zero.

Note that

$$|\Delta PE^{Equity}| = \left(1 - \frac{I}{[\theta^m + d(1 - \frac{2I}{W})](X^H - X^L) + X^L}\right)\frac{2dI}{W}(X^H - X^L). \tag{B104}$$

Since we have assumed that $1 - \frac{2I}{W} > 0$, both $\left(1 - \frac{I}{[\theta^m + d(1 - \frac{2I}{W})](X^H - X^L) + X^L}\right)$ and $\frac{2dI}{W}(X^H - X^L)$ increase in d, we therefore have

$$\frac{\partial |\Delta PE^{Equity}|}{\partial d} > 0, \tag{B105}$$

which means the price impact of equity increases with the dispersion in outsiders' beliefs. Q.E.D.

Proof of Proposition 9

We showed in the proof of proposition 8 that the stock price at issuance is

$$PE_1^{Equity} = \frac{\hat{\theta}X^H + (1 - \hat{\theta})X^L}{1 + \frac{I}{\hat{\theta}X^H + (1-\hat{\theta})X^L - I}}, \tag{B106}$$

where $\hat{\theta} = \theta^m + d\left(1 - \frac{2I}{W}\right)$. The expected stock price at time 2 is

$$E\left[PE_2^{Equity}\right] = \frac{\hat{\theta}_2 X^H + (1 - \hat{\theta}_2)X^L}{1 + \frac{I}{\hat{\theta}X^H + (1-\hat{\theta})X^L - I}}, \tag{B107}$$

where $\hat{\theta}_2 = \theta^m + d(1-\delta)\left(1 - \frac{2I}{W}\right)$. The expected long-run stock return is therefore

$$LR^{Equity} = \frac{E\left[PE_2^{Equity}\right] - PE_1^{Equity}}{PE_1^{Equity}} = \frac{-\delta d(1 - \frac{2I}{W})(X^H - X^L)}{\hat{\theta} X^H + (1 - \hat{\theta})X^L} < 0. \tag{B108}$$

Q.E.D.

Proof of Proposition 10
Partially differentiating LR^{Equity} from (B108), we obtain

$$\frac{\partial LR^{Equity}}{\partial \delta} = \frac{-d(1 - \frac{2I}{W})(X^H - X^L)}{\hat{\theta} X^H + (1 - \hat{\theta})X^L} < 0, \tag{B109}$$

$$\frac{\partial LR^{Equity}}{\partial d} = -\delta\left(1 - \frac{2I}{W}\right)(X^H - X^L)\frac{(\theta^m X^H + (1 - \theta^m)X^L)}{(\hat{\theta} X^H + (1 - \hat{\theta})X^L)^2} < 0. \tag{B110}$$

Q.E.D.

Proof of Proposition 11
The equity price at the time of straight debt issue (time 1) is

$$PE_1^{Debt} = (\theta^m + d)\left(X^H - \frac{I - (1 - \hat{\theta})X^L}{\hat{\theta}}\right), \tag{B111}$$

and the expected equity price at time 2 is

$$E\left[PE_2^{Debt}\right] = (\theta^m + d(1 - \delta))\left(X^H - \frac{I - (1 - \hat{\theta})X^L}{\hat{\theta}}\right). \tag{B112}$$

The expected long-run stock return subsequent to the straight debt issue is therefore

$$LR^{Debt} = \frac{E[PE_2^{Debt}] - PE_1^{Debt}}{PE_1^{Debt}} = \frac{-\delta d}{\theta^m + d} < 0. \tag{B113}$$

For convertible debt, at the time of convertible debt issue (time 1), we have

$$PE_1^{Convertible} = (\theta^m + d)\frac{1}{1 + xF}X^H, \tag{B114}$$

and the expected equity price at time 2 is

$$E\left[PE_2^{Convertible}\right] = (\theta^m + d(1 - \delta))\frac{1}{1 + xF}X^H. \tag{B115}$$

The expected long-run stock return subsequent to the convertible debt issue is therefore

$$LR^{Convertible} = \frac{E[PE_2^{Convertible}] - PE_1^{Convertible}}{PE_1^{Convertible}} = \frac{-\delta d}{\theta^m + d} < 0. \tag{B116}$$

Note that $\frac{\partial LR^{Debt}}{\partial \delta} = \frac{-d}{\theta^m + d} < 0$ and $\frac{\partial LR^{Debt}}{\partial d} = \frac{-\delta\theta^m}{(\theta^m + d)^2} < 0$. Q.E.D.

Proof of Proposition 12

We assume that d is uniformly distributed in the interval $[0, 1 - \theta^m]$. The firm will issue equity at time 1 only when $\theta^f < \hat{\theta}$, that is, when $d > \frac{\theta^f - \theta^m}{1 - \frac{2I}{W}}$. Therefore, the expected long-run stock return of equity issuance is

$$\overline{LR}^{Equity} = \int_{\frac{\theta^f - \theta^m}{1 - \frac{2I}{W}}}^{1-\theta^m} \frac{-\delta x\left(1 - \frac{2I}{W}\right)(X^H - X^L)}{\hat{\theta} X^H + (1 - \hat{\theta})X^L} \cdot \frac{1}{1 - \theta^m - \frac{\theta^f - \theta^m}{1 - \frac{2I}{W}}} dx,$$

$$= \frac{-\delta(X^H - X^L)}{1 - \theta^f - (1 - \theta^m)\frac{2I}{W}} \int_{\frac{\theta^f - \theta^m}{1 - \frac{2I}{W}}}^{1-\theta^m} \frac{x}{X^m + \left(1 - \frac{2I}{W}\right)(X^H - X^L)x} dx,$$

$$\overline{LR}^{Equity} = \frac{-\delta(X^H - X^L)}{1 - \theta^f - (1 - \theta^m)\frac{2I}{W}} \left[\frac{x}{\left(1 - \frac{2I}{W}\right)(X^H - X^L)} - \frac{X^m \ln\left(X^m + \left(1 - \frac{2I}{W}\right)(X^H - X^L)x\right)}{\left(1 - \frac{2I}{W}\right)^2 (X^H - X^L)^2} \right]_{\frac{\theta^f - \theta^m}{1 - \frac{2I}{W}}}^{1-\theta^m},$$

$$= -\frac{\delta}{\left(1 - \frac{2I}{W}\right)} \left[1 - \frac{X^m}{\left(1 - \theta^f - \frac{2I}{W}(1 - \theta^m)\right)(X^H - X^L)} \ln\left(\frac{X^m + \left(1 - \frac{2I}{W}\right)(X^H - X^L)(1 - \theta^m)}{X^m + \left(1 - \frac{2I}{W}\right)(X^H - X^L)\frac{\theta^f - \theta^m}{1 - \frac{2I}{W}}} \right) \right].$$

(B117)

The expected long-run stock return following debt issues are

$$\overline{LR}^{Debt} = \frac{\left(1 - \frac{2I}{W}\right)}{\theta^f - \theta^m} \int_0^{\frac{\theta^f - \theta^m}{\left(1 - \frac{2I}{W}\right)}} \frac{-\delta x}{\theta^m + x} dx = -\frac{\left(1 - \frac{2I}{W}\right)}{\theta^f - \theta^m} \delta[x - \theta^m \ln(\theta^m + x)]_0^{\frac{\theta^f - \theta^m}{\left(1 - \frac{2I}{W}\right)}},$$

$$= -\frac{\left(1 - \frac{2I}{W}\right)}{\theta^f - \theta^m} \delta \left[\frac{\theta^f - \theta^m}{\left(1 - \frac{2I}{W}\right)} - \theta^m \ln\left(1 + \frac{\theta^f - \theta^m}{\theta^m\left(1 - \frac{2I}{W}\right)} \right) \right],$$

(B118)

$$= -\delta \left[1 - \frac{\theta^m\left(1 - \frac{2I}{W}\right)}{\theta^f - \theta^m} \ln\left(1 + \frac{\theta^f - \theta^m}{\theta^m\left(1 - \frac{2I}{W}\right)} \right) \right].$$

Note that $\overline{LR}^{Equity} < \overline{LR}^{Debt}$ if and only if the following parameter condition is satisfied:

$$\frac{1}{1 - \frac{2I}{W}} - 1 + \frac{\theta^m\left(1 - \frac{2I}{W}\right)}{\theta^f - \theta^m} \ln\left(1 + \frac{\theta^f - \theta^m}{\theta^m\left(1 - \frac{2I}{W}\right)} \right) -$$

$$\frac{X^m}{\left(1 - \frac{2I}{W}\right)\left(1 - \theta^f - \frac{2I}{W}(1 - \theta^m)\right)(X^H - X^L)} \ln\left(\frac{X^m + \left(1 - \frac{2I}{W}\right)(X^H - X^L)(1 - \theta^m)}{X^m + \left(1 - \frac{2I}{W}\right)(X^H - X^L)\frac{\theta^f - \theta^m}{1 - \frac{2I}{W}}} \right) > 0. \quad \text{(B119)}$$

Q.E.D.

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